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2018

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citation for published version (APA)

Nijholt, E. C. (2018). *Bifurcations in Network Dynamical Systems*. [PhD-Thesis - Research and graduation internal, Vrije Universiteit Amsterdam].

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CHAPTER 2

GRAPH FIBRATIONS AND SYMMETRIES OF NETWORK DYNAMICS

2.1 Abstract

Dynamical systems with a network structure can display remarkable phenomena such as synchronisation and anomalous synchrony breaking. A methodology for classifying patterns of synchrony in networks was developed by Golubitsky and Stewart. They showed that the robustly synchronous dynamics of a network is determined by its quotient networks. This result was recently reformulated by DeVille and Lerman, who pointed out that the reduction from a network to a quotient is an example of a graph fibration. The current paper exploits this observation and demonstrates the importance of self-fibrations of network graphs. Self-fibrations give rise to symmetries in the dynamics of a network. We show that every network admits a lift with a semigroup or semigroupoid of self-fibrations. The resulting symmetries impact the global dynamics of the network and can therefore be used to explain and predict generic scenarios for synchrony breaking. Also, when the network has a trivial symmetry groupoid, then every robust synchrony in the lift is determined by symmetry.

2.2 Introduction

There are remarkable similarities between dynamical systems with a network structure and dynamical systems with symmetry. For instance, symmetric dynamical systems automatically support symmetric solutions. A specific network structure can analogously force a dynamical system to admit synchronous and partially synchronous solutions [5, 10, 13, 12, 16, 20, 25]. This phenomenon is known as “robust network synchrony”. It is an element that distinguishes the dynamics of networks from that of arbitrary systems. Robust network synchrony is fully understood since the work of Golubitsky and Stewart et al. [12, 16, 24, 25].

In addition, network dynamical systems can display very unusual bifurcations [1, 4, 8, 9, 10, 11, 17, 18, 22]. In particular, there are many examples of network systems with anomalous steady state and Hopf bifurcations. These so-called *synchrony breaking bifurcations* are often governed by spectral degeneracies and are reminiscent of the symmetry breaking bifurcations that occur in equivariant (symmetric) dynamical systems. We note that the presence of robust synchrony does not explain these anomalous bifurcations, because synchrony does not affect the global phase space of a network. What causes the anomalous synchrony breaking of networks remains unknown in general.

Golubitsky and Stewart et al. [12, 16, 24, 25] showed that the robustly synchronous dynamics of a network is determined by its so-called “quotient networks”. Robust synchrony can therefore completely be understood in terms of the network graph. Quotient networks arise by identifying the cells of the original network that evolve synchronously. DeVille and Lerman [3] recently pointed out that the corresponding quotient map (from the original network graph to its quotient) is an example of a so-called “graph fibration”. They also show that every graph fibration $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ between two networks produces a conjugacy between the dynamics of \mathbf{N}_2 and the dynamics \mathbf{N}_1 . This latter fact had also been proved by computer scientists [2] in 2002. The result of DeVille and Lerman can be thought of as a geometric reformulation of the original result of Golubitsky and Stewart.

There are many classical tools for the study and classification of equivariant bifurcations and equivariant singularities [7, 6, 14, 15]. These include representation theory and equivariant singularity theory. It is quite disappointing that similar “tailored” techniques currently do not exist for the analysis of bifurcations in networks. The reason, arguably, is that network structure is not an intrinsic geometric property of a dynamical system: it is not preserved under coordinate changes.

A first step towards overcoming this problem will be taken in this paper.

We shall use the language of graph fibrations to describe a new geometric invariant of networks with obvious dynamical consequences. Our main result is based on the simple observation that every self-fibration (i.e. graph fibration from a network graph to itself) yields a symmetry in the dynamics of a network. Self-fibrations should therefore be thought of as symmetries of network graphs. The self-fibrations of a network in general do not form a group but a semigroup, so self-fibrations may not correspond to classical symmetries. Network graphs also need not admit any nontrivial self-fibrations, but we will prove that every network is a quotient of a network with self-fibrations. In fact, we prove the following theorem, that will later be formulated more precisely.

Theorem 2.2.1. *Every network \mathbf{N} is (in a natural way) the quotient of a network $\tilde{\mathbf{N}}$ that admits a semigroup or semigroupoid $\Sigma_{\mathbf{N}}$ of self-fibrations. The dynamics of \mathbf{N} is therefore embedded in that of $\tilde{\mathbf{N}}$. Moreover, the dynamics of $\tilde{\mathbf{N}}$ is $\Sigma_{\mathbf{N}}$ -equivariant.*

Theorem 3.2.1 shows that every network dynamical system can be thought of as a dynamical system with “hidden symmetry”. This hidden symmetry in general consists of a semigroup or semigroupoid (rather than a group) and need not act on the network itself, but on a “lift” of it. We will show that hidden symmetry can put severe geometric restrictions on the dynamics of a network, and that it may have a nontrivial impact on the singularities that determine the emergence and breaking of synchrony.

Theorem 3.2.1 implies in particular that networks can be studied with techniques that are common in equivariant dynamics. For example, hidden symmetry can be accommodated for in standard ODE methods such as Lyapunov-Schmidt reduction, center manifold reduction and Poincaré normal forms. Hidden symmetry also provides a possibility to classify stable singularities and bifurcations in networks, using representation theory and equivariant singularity theory. None of these techniques and methods currently exist for network systems. We discuss this point in some detail in Section 2.9.

As a byproduct of Theorem 3.2.1, we will also obtain:

Corollary 2.2.2. *When \mathbf{N} does not have interchangeable inputs (i.e. all inputs that its cells receive are distinct), then every robust synchrony in the lift $\tilde{\mathbf{N}}$ (and hence every robust synchrony in \mathbf{N} itself) is an invariant subspace for any $\Sigma_{\mathbf{N}}$ -equivariant dynamical system.*

This means that robust synchrony is often itself a consequence of hidden symmetry.

This paper is based on ideas that are present in rudimentary form in our earlier work [23], but we extend these ideas here and formulate them in a more geometric language. We moreover demonstrate now that hidden symmetry has far-reaching consequences for dynamics.

This paper is organised as follows. We start by discussing a few remarkable phenomena in network dynamical systems in Section 2.3. We review some existing general theory on coupled cell networks in Sections 2.4 and 2.5. So-called “homogeneous” networks and their symmetry properties are studied in Sections 2.6, 2.7 and 2.8, and we prove Theorem 3.2.1 for these networks in Section 2.8. Finally, in Section 2.9 we discuss the importance of hidden network symmetry, and we demonstrate how it can be used in the study of local bifurcations.

Acknowledgement

The authors would like to thank Marty Golubitsky for his interest in this topic, and for asking the first questions that eventually led to this paper.

2.3 Three Examples

Figure 2.1 depicts the homogeneous networks **A**, **B** and **C** (see Section 2.6 for a definition) that each contain three vertices receiving two different arrows. One could think of these networks as consisting of (groups of) identical neurons, that each receive for instance one excitatory signal (say through the solid blue arrow) and one inhibitory signal (the dashed red arrow). The states of the cells of the networks are determined by variables

$x_{v_1}, x_{v_2}, x_{v_3} \in \mathbb{R}$ (for example membrane potentials). These variables then obey the equations of motion displayed below the network graphs in Figure 2.1. The response function $f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ describes the precise dependence of the evolution of each cell on its own state and on its two incoming signals, and thus determines the actual dynamics of the network. We let this f depend on a parameter $\lambda \in \mathbb{R}$, to express that it can sometimes be modified in experiments, or that it may not be entirely known.

In spite of their different architectures, the dynamics of networks **A**, **B** and **C** admit exactly the same (partial) synchronies. For example, setting $x_{v_1} = x_{v_2}$ in the equations of motion of either one of the networks, yields that $\dot{x}_{v_1} = \dot{x}_{v_2}$. The subspace $\{x_{v_1} = x_{v_2}\}$ is thus invariant under the dynamics of all three network systems, independently of the precise form of the function f . Similarly, $x_{v_1} = x_{v_2} = x_{v_3}$ gives that $\dot{x}_{v_1} = \dot{x}_{v_2} = \dot{x}_{v_3}$. Moreover, these are the only such equalities. We conclude that the subspaces

$\{x_{v_1} = x_{v_2}\}$ (“partial synchrony”) and $\{x_{v_1} = x_{v_2} = x_{v_3}\}$ (“full synchrony”)

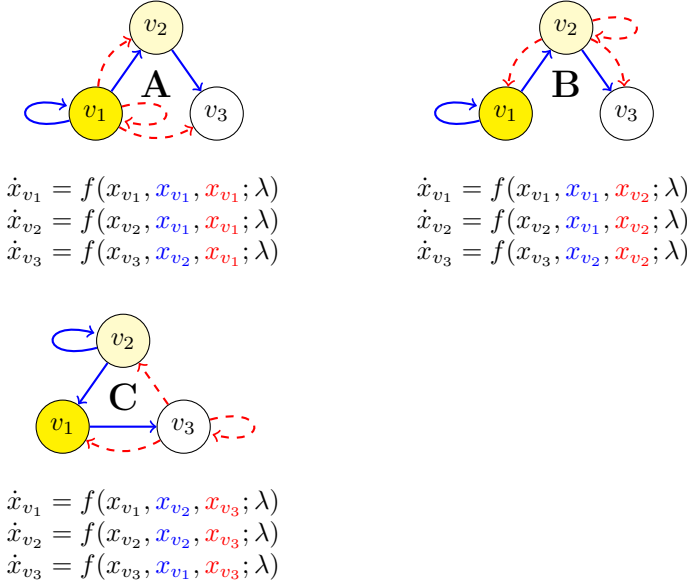


Figure 2.1: Homogeneous networks with 3 identical cells and 2 types of inputs.

are the two “robust synchronies” of networks **A**, **B** and **C**. They can be thought of as dynamical invariants of the network graphs, see Section 2.5.

To understand how synchrony can emerge or disappear, assume now that $f(0, 0, 0; \lambda) = 0$. This means that $x = (0, 0, 0)$ is a fully synchronous steady state of the network dynamics for all values of the parameter λ . One then says that a “synchrony breaking steady state bifurcation” occurs at $\lambda = 0$, when less synchronous steady states emerge near this fully synchronous state as λ varies near 0. This can only happen if for $\lambda = 0$, the linearisation matrix of the differential equations around $x = (0, 0, 0)$ is degenerate. This linearisation matrix is easy to compute and reads (writing $a = D_1 f(0, 0, 0; 0)$ etc.)

$$\begin{aligned}
\text{for network } \mathbf{A}: & \begin{pmatrix} a + \textcolor{blue}{b} + \textcolor{red}{c} & 0 & 0 \\ & \textcolor{blue}{b} + \textcolor{red}{c} & a \\ & & \textcolor{red}{c} & \textcolor{blue}{b} & a \end{pmatrix}; \\
\text{for network } \mathbf{B}: & \begin{pmatrix} a + \textcolor{blue}{b} & & \textcolor{red}{c} & 0 \\ & \textcolor{blue}{b} & a + \textcolor{red}{c} & 0 \\ & 0 & \textcolor{blue}{b} + \textcolor{red}{c} & a \end{pmatrix}; \\
\text{for network } \mathbf{C}: & \begin{pmatrix} a & & \textcolor{blue}{b} & \textcolor{red}{c} \\ 0 & a + \textcolor{blue}{b} & & \textcolor{red}{c} \\ & \textcolor{blue}{b} & 0 & a + \textcolor{red}{c} \end{pmatrix}.
\end{aligned}$$

Interestingly, these three linearisation matrices all have an eigenvalue $a + \textcolor{blue}{b} + \textcolor{red}{c}$ with multiplicity 1 and a defective eigenvalue a with algebraic multiplicity 2 and geometric multiplicity 1. The eigenvector for the eigenvalue $a + \textcolor{blue}{b} + \textcolor{red}{c}$ is fully synchronous, so synchrony breaking can only occur when $a = 0$. The degeneracy of this eigenvalue suggests that the resulting steady state bifurcation may be quite unusual. Indeed, a singularity analysis (that we have not included here) reveals that in a generic one-parameter synchrony breaking bifurcation in any one of the networks, two branches of steady states $x(\lambda)$ are born from the synchronous state: a partially synchronous and a non-synchronous branch. Table 2.1 shows the asymptotics of these branches for the three networks.

Although networks **A**, **B** and **C** display exactly the same robust synchronies and spectral properties, Table 2.1 reveals that their synchrony breaking bifurcations are very different. For example, the generic synchrony breaking branches of the three networks clearly have different asymptotics. Another distinction between the networks concerns the dynamical stability of the bifurcating branches. It turns out that in a generic synchrony breaking bifurcation, the fully synchronous state loses stability when λ passes through 0. It can be shown that in networks **A** and **B**, it is then only possible that the non-synchronous state becomes stable, but it turns out that in network **C**, stability can also be transferred to the partially synchronous state.

The different synchrony breaking behaviour of networks **A**, **B** and **C** depends on nonlinearities in the differential equations. In this paper we shall show that this nonlinear effect is fully determined by hidden symmetry.

2.4 Networks

Every dynamical system trivially has a network structure. Nevertheless, the observables of certain dynamical systems have a nontrivial interaction struc-

Network A

Asymptotics	Synchrony
$x_{v_1} = x_{v_2} = x_{v_3} = 0$	Full
$x_{v_1} = x_{v_2} = 0, x_{v_3} \sim \lambda$	Partial
$x_{v_1} = 0, x_{v_2} \sim \lambda, x_{v_3} \sim \pm\sqrt{\lambda}$	None

Network B

Asymptotics	Synchrony
$x_{v_1} = x_{v_2} = x_{v_3} = 0$	Full
$x_{v_1} = x_{v_2} = 0, x_{v_3} \sim \lambda$	Partial
$x_{v_1} \sim \lambda, x_{v_2} \sim \lambda$ $x_{v_1} - x_{v_2} \sim \lambda, x_{v_3} \sim \pm\sqrt{\lambda}$	None

Network C

Asymptotics	Synchrony
$x_{v_1} = x_{v_2} = x_{v_3} = 0$	Full
$x_{v_1} = x_{v_2} \sim \lambda, x_{v_3} \sim \lambda, x_{v_{1,2}} - x_{v_3} \sim \lambda$	Partial
$x_{v_1} \sim \lambda, x_{v_2} \sim \lambda, x_{v_3} \sim \lambda$ $x_{v_1} - x_{v_2} \sim \lambda^2, x_{v_{1,2}} - x_{v_3} \sim \lambda$	None but almost partial

Table 2.1: Asymptotics of steady state branches in generic synchrony breaking bifurcations.

ture. Such a structure can be encoded in a network graph, that describes how the evolution of each observable depends on the values of others. In the literature [3, 12, 16], these network graphs or “coupled cell networks” are usually finite directed graphs, of which the vertices (also referred to as “cells”) and arrows (“couplings”) are all of a certain specified type (“color”). One has in mind that every cell has a state, that evolves in time under the influence of those cells from which it receives a coupling. One also asks for compatibility between the colored cells and colored couplings, to express that cells of the same type respond in the same way to their inputs. We will use the following definition in this paper:

Definition 2.4.1. A *network* or *coupled cell network* is a finite directed graph $\mathbf{N} = \{A \rightrightarrows_t^s V\}$ (where A are the arrows, V are the vertices, and s and t denote the source and target maps), in which all vertices and arrows are assigned a color (chosen from some finite set of colors), such that

1. if two arrows $a_1, a_2 \in A$ have the same color, then so do their sources $s(a_1)$ and $s(a_2)$, and so do their targets $t(a_1)$ and $t(a_2)$.

2. if two vertices $v_1, v_2 \in V$ have the same color, then there is a color-preserving bijection $\beta_{v_2, v_1} : t^{-1}(v_1) \rightarrow t^{-1}(v_2)$ between the arrows that target v_1 and v_2 .

We remark here that the collection of color preserving bijections

$$\mathbb{G} := \{ \beta_{v_2, v_1} : t^{-1}(v_1) \rightarrow t^{-1}(v_2) \text{ color preserving bijection} \mid v_1, v_2 \in V \}$$

is called the *symmetry groupoid* of the network \mathbf{N} . This terminology is due to Golubitsky, Stewart and Pivato [25]. The set \mathbb{G} is a groupoid, because its elements are invertible and the compositions $\beta_{v_3, v_2} \circ \beta_{v_2, v_1}$ define a partial associative product. The symmetry groupoid describes “local symmetries” or “input equivalences” between cells. Indeed, for fixed vertices $v_1, v_2 \in V$, we can define $\mathbb{G}_{v_2, v_1} := \{ \beta_{v_2, v_1} \in \mathbb{G} \}$. By Definition 2.4.1, this set is nonempty if and only if v_1 and v_2 have the same color, and this implies that the so-called “vertex groups” \mathbb{G}_{v_1, v_1} and \mathbb{G}_{v_2, v_2} are then isomorphic.

Following [12, 25], we now describe a natural class of dynamical systems that are compatible with a given network \mathbf{N} . These network dynamical systems are determined by so-called *admissible maps* or *network maps*. To define these, we will first of all assume that every vertex $v \in \mathbf{N}$ has a “state” determined by a variable $x_v \in E_v$, taking values in a finite-dimensional vector space E_v (or a manifold, but we will not pursue this straightforward generalisation). The total state of the network is thus given by an element

$$x \in E_{\mathbf{N}} := \prod_{v \in V} E_v.$$

We have in mind that the state x_v of cell v evolves under the influence of only those x_w for which there is an arrow $a \in A$ with $s(a) = w$ and $t(a) = v$. This inspires us to choose, for every vertex $v \in V$, a “response function”

$$f^v : \prod_{t(a)=v} E_{s(a)} \rightarrow E_v.$$

Note that f^v may depend on certain state variables x_w repeatedly, if different arrows $a_1 \neq a_2$ that both target v have the same source w .

The network and response functions together now yield a map with a “network structure”:

$$\gamma_f^{\mathbf{N}} : E_{\mathbf{N}} \rightarrow E_{\mathbf{N}} \text{ defined by } (\gamma_f^{\mathbf{N}})_v(x) := f^v \left(\prod_{t(a)=v} x_{s(a)} \right).$$

As required, $(\gamma_f^{\mathbf{N}})_v(x)$ only depends on the variables $x_{s(a)}$ for those $a \in A$ with $t(a) = v$.

Finally, we want to impose some restrictions on the response functions to ensure compatibility of $\gamma_f^{\mathbf{N}}$ with the coloring of the arrows and vertices of the network. First of all, it is natural to assume that vertices with the same color have the same sets of state variables:

$$E_{v_1} = E_{v_2} \text{ when } v_1 \text{ and } v_2 \text{ have the same color.}$$

It then follows from Definition 2.4.1 that $E_{s(a)} = E_{s(\beta_{v_2, v_1}(a))}$ for all $a \in A$ with $t(a) = v_1$. This last observation allows us to define, for each $\beta_{v_2, v_1} \in \mathbb{G}$, an input identification map

$$\beta_{v_2, v_1}^* : \prod_{t(a)=v_2} E_{s(a)} \rightarrow \prod_{t(a)=v_1} E_{s(a)} \text{ by } (\beta_{v_2, v_1}^* X)_{s(a)} := X_{s(\beta_{v_2, v_1}(a))}.$$

We shall require that cells of the same color respond in the same way to their incoming signals, and that signals of the same color have the same impact on a cell, i.e.

3. the response functions are groupoid-invariant:

$$f^{v_1} \circ \beta_{v_2, v_1}^* = f^{v_2} \text{ for all } \beta_{v_2, v_1} \in \mathbb{G}.$$

This final assumption expresses how the local symmetries of the network \mathbf{N} give rise to local symmetries in the components of the network maps $\gamma_f^{\mathbf{N}}$. In particular, if a vertex group $\mathbb{G}_{v, v}$ is nontrivial, then f^v must be invariant under certain permutations of inputs.

We summarise our assumptions in the following definition:

Definition 2.4.2. A map $\gamma : E_{\mathbf{N}} \rightarrow E_{\mathbf{N}}$ is a *network map* or *admissible map* for $\mathbf{N} = \{A \rightrightarrows_t^s V\}$ if there is a set of smooth response functions $\{f^v\}_{v \in V}$ satisfying 3, so that $\gamma = \gamma_f^{\mathbf{N}}$.

A “network dynamical system” is now given by the flow of the ordinary differential equation

$$\dot{x} = \gamma_f^{\mathbf{N}}(x) \text{ with } x \in E_{\mathbf{N}}.$$

As was pointed out in [3], one may think of this ODE as a set of coupled “open control systems” (namely the individual ODEs $\dot{x}_v = f^v \left(\prod_{t(a)=v} x_{s(a)} \right)$ for $v \in V$). Rather than ODEs, one may also consider discrete-time network dynamical systems on $E_{\mathbf{N}}$ of the form

$$x_{n+1} = \gamma_f^{\mathbf{N}}(x_n).$$

We conclude by remarking that, as in Section 2.3, we sometimes want to study parameter families of network dynamical systems. Then the response functions $f^v = f^v(\cdot; \lambda)$ themselves become smooth functions of a parameter λ that takes values in some open set $\Lambda \subset \mathbb{R}^p$. For the moment, we shall suppress this parameter dependence in our notation though.

2.5 Graph Fibrations and Robust Synchrony

Synchrony and partial synchrony are prominent forms of collective behaviour of network systems, in which certain cells undergo the same evolution. In this section, we briefly summarise the characterisations of robust synchrony as given by Golubitsky and Stewart [25] on the one hand and DeVille and Lerman [3] on the other. Mathematically, synchrony can be described as follows. Let $P = \{P_1, \dots, P_r\}$ be a partition of the cells of a network $\mathbf{N} = \{A \rightrightarrows_t^s V\}$, that is $P_1 \cup \dots \cup P_r = V$, and $P_i \cap P_j = \emptyset$ if $i \neq j$. For $v_1, v_2 \in V$, we shall write $v_1 \sim_P v_2$ if there is a k such that $v_1, v_2 \in P_k$. We then define the *polydiagonal subspace* or *synchrony subspace* $\text{Syn}_P \subset E_{\mathbf{N}}$ associated to this partition as

$$\text{Syn}_P := \{x \in E_{\mathbf{N}} \mid x_{v_1} = x_{v_2} \text{ when } v_1 \sim_P v_2\}.$$

For this definition to make sense, one must of course require that $E_{v_1} = E_{v_2}$ when $v_1 \sim_P v_2$.

Of dynamical interest are those synchronies that are preserved in time. Such synchronies are determined by polydiagonal subspaces that are invariant under the network dynamics, i.e. for which $\gamma_f^{\mathbf{N}}(\text{Syn}_P) \subset \text{Syn}_P$. This latter inclusion clearly depends on the choice of the response functions f^v , but certain synchrony subspaces are always dynamically invariant, irrespective of the choice of response functions. These special synchrony subspaces are called “robust”. They depend only on the network \mathbf{N} .

The following well-known result characterises the robust synchrony subspaces in terms of the network structure. For a proof of Theorem 2.5.1, we refer to [25].

Theorem 2.5.1. *Let P be a partition of the cells of a network \mathbf{N} . The following are equivalent:*

- i) $\gamma_f^{\mathbf{N}}(\text{Syn}_P) \subset \text{Syn}_P$ for all choices of response functions $\{f^v\}_{v \in V}$ satisfying **3**. In this case, one says that Syn_P is a *robust synchrony subspace*.
- ii) For all vertices $v_1 \sim_P v_2$, there is a $\beta_{v_2, v_1} \in \mathbb{G}_{v_2, v_1}$ such that for every arrow a with $t(a) = v_1$, it holds that $s(a) \sim_P s(\beta_{v_2, v_1}(a))$. The partition is then called *balanced*.

Theorem 2.5.1 was recently generalized and reformulated by DeVille and Lerman [3]. They formulate their result in the language of category theory and graph fibrations.

Definition 2.5.2. A map $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ of networks is a *graph fibration* if

- i) it sends cells to cells of the same color, arrows to arrows of the same color, and the head and tail of every arrow $a_1 \in \mathbf{N}_1$ to the head and tail of $\phi(a_1) \in \mathbf{N}_2$;
- ii) for every cell $v_1 \in \mathbf{N}_1$ and every arrow $a_2 \in \mathbf{N}_2$ ending at $\phi(v_1)$, there is a unique arrow $a_1 \in \phi^{-1}(a_2)$ that ends at v_1 .

Property i) simply requires that ϕ is a morphism of colored directed graphs. Property ii) is the fibration property: it says that ϕ restricts to a color-preserving bijection

$$\phi|_{t^{-1}(v_1)} : t^{-1}(v_1) \rightarrow t^{-1}(\phi(v_1)),$$

between the arrows targeting any vertex $v_1 \in \mathbf{N}_1$ and those targeting its image $\phi(v_1) \in \mathbf{N}_2$.

When $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ is a graph fibration, we call \mathbf{N}_2 a *quotient* of \mathbf{N}_1 and \mathbf{N}_1 a *lift* of \mathbf{N}_2 . Despite this terminology, we will not require that ϕ is surjective. Figure 2.2 depicts quotients of networks **A**, **B** and **C**, and the action of the corresponding graph fibrations on vertices.

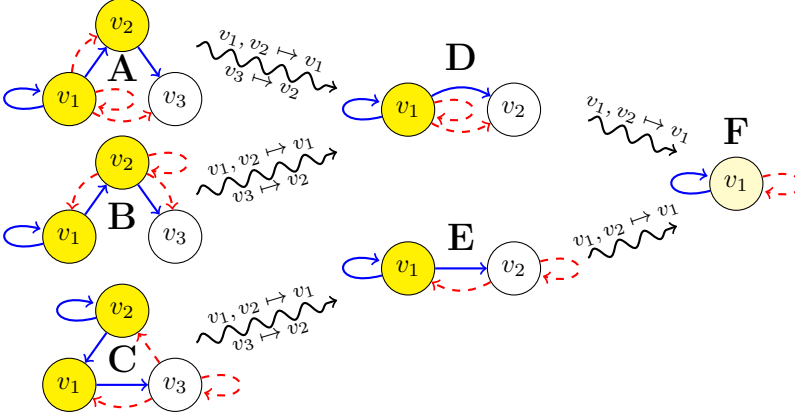


Figure 2.2: Graph fibrations that explain the robust synchronies of networks **A**, **B** and **C**.

The dynamical relevance of graph fibrations is explained by the following theorem, a proof of which is given in [3].

Theorem 2.5.3 (DeVillegger & Lerman). *Let $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ be a graph fibration. Define the map $\phi^* : E_{\mathbf{N}_2} \rightarrow E_{\mathbf{N}_1}$ between the phase spaces of networks \mathbf{N}_2 and \mathbf{N}_1 by*

$$(\phi^* y)_v := y_{\phi(v)}.$$

Then ϕ^* sends every solution $y(t)$ of the dynamics of network \mathbf{N}_2 to a solution $x(t) := \phi^*y(t)$ of the dynamics of network \mathbf{N}_1 , that is

$$\phi^* \circ \gamma_f^{\mathbf{N}_2} = \gamma_f^{\mathbf{N}_1} \circ \phi^*.$$

The solution $x(t) = \phi^*y(t)$ has the robust synchrony $x_{v_1}(t) = x_{v_2}(t)$ when $\phi(v_1) = \phi(v_2)$. Moreover, every robust synchrony of \mathbf{N}_1 arises from a graph fibration in this way.

Example 2.5.4. Consider the graph fibration $\phi : \mathbf{A} \rightarrow \mathbf{D}$ in Figure 2.2 that maps cells $v_1, v_2 \in \mathbf{A}$ to $v_1 \in \mathbf{D}$ and $v_3 \in \mathbf{A}$ to $v_2 \in \mathbf{D}$. The resulting conjugacy is the map

$$\phi^*(y_{v_1}, y_{v_2}) = (y_{v_1}, y_{v_1}, y_{v_2}).$$

Indeed, if $(y_{v_1}(t), y_{v_2}(t))$ is a solution of the equations of network \mathbf{D} ,

$$\begin{aligned} \dot{y}_{v_1} &= f(y_{v_1}, \textcolor{blue}{y}_{v_1}, \textcolor{red}{y}_{v_1}; \lambda) \\ \dot{y}_{v_2} &= f(y_{v_2}, \textcolor{blue}{y}_{v_1}, \textcolor{red}{y}_{v_1}; \lambda) \end{aligned} ,$$

then $(x_{v_1}(t), x_{v_2}(t), x_{v_3}(t)) := \phi^*(y_{v_1}(t), y_{v_2}(t)) = (y_{v_1}(t), y_{v_1}(t), y_{v_2}(t))$ is a partially synchronous solution for network \mathbf{A} , that is

$$\begin{aligned} \dot{x}_{v_1} &= f(x_{v_1}, \textcolor{blue}{x}_{v_1}, \textcolor{red}{x}_{v_1}; \lambda) \\ \dot{x}_{v_2} &= f(x_{v_2}, \textcolor{blue}{x}_{v_1}, \textcolor{red}{x}_{v_1}; \lambda) \\ \dot{x}_{v_3} &= f(x_{v_3}, \textcolor{blue}{x}_{v_2}, \textcolor{red}{x}_{v_1}; \lambda) \end{aligned} .$$

Similarly for the other graph fibrations in Figure 2.2, that are responsible for the robust synchronies of the networks \mathbf{A} , \mathbf{B} and \mathbf{C} that were discussed in Section 2.3 .

More so than the rather combinatorial Theorem 2.5.1, Theorem 2.5.3 provides a geometric explanation of the occurrence of synchrony: robust synchrony is a consequence of the existence of graph fibrations and of the resulting conjugacies of dynamical systems.

Remark 2.5.5. Let $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ and $\psi : \mathbf{N}_2 \rightarrow \mathbf{N}_3$ be graph fibrations and

$$\phi^* : E_{\mathbf{N}_2} \rightarrow E_{\mathbf{N}_1} \text{ and } \psi^* : E_{\mathbf{N}_3} \rightarrow E_{\mathbf{N}_2}$$

the conjugacies resulting from Theorem 2.5.3. Then $\psi \circ \phi : \mathbf{N}_1 \rightarrow \mathbf{N}_3$ is also a graph fibration. Moreover, for $z \in E_{\mathbf{N}_3}$ we have $((\psi \circ \phi)^*z)_v = z_{(\psi \circ \phi)(v)} = z_{\psi(\phi(v))} = (\psi^*z)_{\phi(v)} = (\phi^*(\psi^*z))_v$. This proves that

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* .$$

Alternatively, one may express this by saying that the map $*$: $\phi \mapsto \phi^*$ determines a contravariant functor from the category of networks to the category of dynamical systems. We refer to [3] for more details on the categorical approach to network dynamics (that we will not use any further in this paper).

The following simple remark will be useful for us later:

Proposition 2.5.6. *When $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ is surjective, then $\phi^* : E_{\mathbf{N}_2} \rightarrow E_{\mathbf{N}_1}$ is injective. When ϕ is injective, then ϕ^* is surjective.*

Proof. This all follows directly from the definition $(\phi^*x)_v := x_{\phi(v)}$.

Assume for instance that $\phi^*y = \phi^*Y$. Then $y_{\phi(v)} = Y_{\phi(v)}$ for all cells v of \mathbf{N}_1 . When ϕ is surjective, this implies that $y_w = Y_w$ for all cells w of \mathbf{N}_2 . Thus, $y = Y$ and ϕ^* is injective.

When ϕ is injective, let $x \in E_{\mathbf{N}_1}$ be given and choose any $y \in E_{\mathbf{N}_2}$ satisfying $y_w = x_v$ whenever $w = \phi(v)$. Injectivity makes this possible. Then $\phi^*y = x$ and ϕ^* is surjective. \square

2.6 Homogeneous Networks

We shall restrict our attention to a rather simple class of networks for a while:

Definition 2.6.1. A *homogeneous network* is a network with vertices of one single color, in which the arrows that target one vertex all have a different color.

A network $\mathbf{N} = \{A \rightrightarrows_t^s V\}$ is homogeneous precisely if $\sharp \mathbb{G}_{v_2, v_1} = 1$ for all pairs $v_1, v_2 \in V$. In particular, every cell of such a network has the same phase space $E_v = E$ and responds in the same way to its incoming signals. Also, signals of a different color may have a different effect on a cell. Homogeneous networks have the advantage that they allow for a rather simple algebraic treatment. For example, one calls the number of incoming arrows of a cell the “valency” or “in-degree” of that cell. Note that homogeneous networks have a single valency. A homogeneous network of valency m can thus conveniently be described by m “input maps”

$$\sigma_1, \dots, \sigma_m : V \rightarrow V,$$

in which $\sigma_j(v)$ is the source of the unique arrow of color j that targets vertex v . It is also clear that a response function is simply a function $f : E^m \rightarrow E$ and that a homogeneous network map $\gamma_f^{\mathbf{N}} : E_{\mathbf{N}} \rightarrow E_{\mathbf{N}}$ must be of the form

$$(\gamma_f^{\mathbf{N}})_v(x) = f(x_{\sigma_1(v)}, \dots, x_{\sigma_m(v)}) \text{ for all } v \in V. \quad (2.6.1)$$

To guarantee that every cell notices its own state, we shall assume from now on that

$$\sigma_1 = \text{Id}_V.$$

Formula (2.6.1) moreover shows that, without loss of generality, we can assume that all the σ_j 's are different: if $\sigma_i = \sigma_j$ for $i \neq j$, then the arrows of colors i and j can be identified, and f can then be redefined to depend on less variables.

Example 2.6.2. Networks **A**, **B** and **C** of Figure 2.1 are examples of homogeneous networks with 3 cells of valency 3. The maps $\sigma_1, \sigma_2, \sigma_3$ are given in this case by

A	v_1	v_2	v_3	B	v_1	v_2	v_3	C	v_1	v_2	v_3
σ_1	v_1	v_2	v_3	σ_1	v_1	v_2	v_3	σ_1	v_1	v_2	v_3
σ_2	v_1	v_1	v_2	σ_2	v_1	v_1	v_2	σ_2	v_2	v_2	v_1
σ_3	v_1	v_1	v_1	σ_3	v_2	v_2	v_2	σ_3	v_3	v_3	v_3

Thus, σ_1 = "arrows from every cell to itself", σ_2 = "all blue arrows" and σ_3 = "all red arrows". Figure 2.1 does not depict the arrows corresponding to σ_1 . The figure also displays the homogeneous network differential equations $\dot{x} = \gamma_f^{\mathbf{A}}(x)$, $\dot{x} = \gamma_f^{\mathbf{B}}(x)$ and $\dot{x} = \gamma_f^{\mathbf{C}}(x)$.

The following simple proposition characterises graph fibrations of homogeneous networks.

Proposition 2.6.3. *Let $\mathbf{N}_1 = \{A_1 \rightrightarrows_{t_1}^{s_1} V_1\}$ and $\mathbf{N}_2 = \{A_2 \rightrightarrows_{t_2}^{s_2} V_2\}$ be homogeneous networks of valency m , respectively with input maps*

$$\sigma_1^{(1)}, \dots, \sigma_m^{(1)} : V_1 \rightarrow V_1 \text{ and } \sigma_1^{(2)}, \dots, \sigma_m^{(2)} : V_2 \rightarrow V_2.$$

Then $\phi : \mathbf{N}_1 \rightarrow \mathbf{N}_2$ is a graph fibration if and only if

$$\phi|_{V_1} \circ \sigma_j^{(1)} = \sigma_j^{(2)} \circ \phi|_{V_1} \text{ for all colors } 1 \leq j \leq m.$$

Proof. It is obvious from Definitions 2.5.2 and 2.6.1 that ϕ must send $\sigma_j^{(1)}(v)$ to $\sigma_j^{(2)}(\phi(v))$. Moreover, it is a graph fibration if it does so. \square

Remark 2.6.4. It is not hard to prove (see for example Proposition 7.2 in [21]) that a partition $V = P_1 \cup \dots \cup P_r$ of the cells of a homogeneous network \mathbf{N} with input maps $\sigma_1, \dots, \sigma_m$ is balanced if and only if for all $1 \leq j \leq m$ and $1 \leq k \leq r$ there is an $1 \leq l \leq r$ such that

$$\sigma_j(P_k) \subset P_l.$$

The input maps then descend to maps on the partition. In fact, we can construct a new homogeneous network \mathbf{N}^P with cells $\{v_1^P, \dots, v_r^P\}$ and input maps $\sigma_1^P, \dots, \sigma_m^P$ that satisfy

$$\sigma_j^P(v_k^P) = v_l^P \text{ if and only if } \sigma_j(P_k) \subset P_l.$$

By definition, the map of vertices

$$\phi : V \rightarrow \{v_1^P, \dots, v_r^P\} \text{ defined by } \phi(v) = v_j^P \text{ if and only if } v \in P_j$$

then satisfies $\phi \circ \sigma_j = \sigma_j^P \circ \phi$ for all $1 \leq j \leq m$ and thus extends to a graph fibration. This confirms that \mathbf{N}^P is a quotient of \mathbf{N} .

“Nonhomogeneous networks”, which have different cell types but no interchangeable inputs, can be described in a similar way [21]. Although the notation is heavier, all the results of this paper remain true for such networks. Networks with a nontrivial symmetry groupoid, in which certain cells receive several arrows of the same color, can not be described by a unique collection of input maps. Some results in this paper therefore do not have an obvious generalisation to such networks.

2.7 The Fundamental Network

In this section we define, for every homogeneous network, the lift with self-fibrations mentioned in the introduction. Recall that every homogeneous network $\mathbf{N} = \{A \rightrightarrows_t^s V\}$ can be described by input maps $\sigma_1, \dots, \sigma_m : V \rightarrow V$, where m is the valency of the network. In general, the composition $\sigma_j \circ \sigma_k$ need not be equal to any σ_i , but there does exist a smallest collection

$$\Sigma_{\mathbf{N}} = \{\sigma_1, \dots, \sigma_m, \dots, \sigma_n\}$$

that contains $\sigma_1, \dots, \sigma_m$ and is closed under composition. By definition, $\Sigma_{\mathbf{N}}$ is the unique semigroup (composition of maps being the semigroup operation) with unit (i.e. $\Sigma_{\mathbf{N}}$ is a “monoid”) generated by $\sigma_1, \dots, \sigma_m$. Note in particular that $\Sigma_{\mathbf{N}}$ is finite, as there are only finitely many maps from the finite set of vertices V to itself.

Remark 2.7.1. One aspect of the relevance of $\Sigma_{\mathbf{N}}$ is easy to explain. Let $1 \leq j_1, \dots, j_q \leq m$ be a sequence of colors. Then there is a path in \mathbf{N} from cell $(\sigma_{j_1} \circ \dots \circ \sigma_{j_q})(v)$ to cell v , consisting of a sequence of arrows of colors j_1, \dots, j_q respectively. Cell $(\sigma_{j_1} \circ \dots \circ \sigma_{j_q})(v)$ thus acts “indirectly” as an input of cell v . Because $\Sigma_{\mathbf{N}}$ is closed under composition, the set

$$V_{(v)} := \{\sigma_j(v) \mid \sigma_j \in \Sigma_{\mathbf{N}}\}$$

is equal to the set of vertices in \mathbf{N} from which there is a path to v . Moreover, $\Sigma_{\mathbf{N}}$ determines all the sets $V_{(v)}$ (with $v \in V$) simultaneously. Nevertheless, it will become clear that much more information is contained in the product structure of $\Sigma_{\mathbf{N}}$.

We are now ready to define another homogeneous network as follows:

Definition 2.7.2. Let \mathbf{N} be a homogeneous network with input maps $\sigma_1, \dots, \sigma_m$ and let $\Sigma_{\mathbf{N}}$ be the above semigroup. The *fundamental network* $\tilde{\mathbf{N}}$ of \mathbf{N} is the homogeneous network with vertex set $\Sigma_{\mathbf{N}}$ and input maps $\tilde{\sigma}_1, \dots, \tilde{\sigma}_m$ defined by

$$\tilde{\sigma}_k(\sigma_j) := \sigma_k \circ \sigma_j \text{ for } 1 \leq k \leq m.$$

In other words, $\tilde{\mathbf{N}}$ contains an arrow of color k from σ_i to σ_j if and only if $\sigma_i = \sigma_k \circ \sigma_j$.

The map $\tilde{\sigma}_k : \Sigma_{\mathbf{N}} \rightarrow \Sigma_{\mathbf{N}}$ encodes the left-multiplicative behaviour of $\sigma_k \in \Sigma_{\mathbf{N}}$. Thus, the fundamental network is a graphical representation of the semigroup $\Sigma_{\mathbf{N}}$ together with its generators $\sigma_1, \dots, \sigma_m$. Such a graphical representation is called a *Cayley graph*. Note that the fundamental network $\tilde{\mathbf{N}}$ of \mathbf{N} can easily be constructed from the product table of $\Sigma_{\mathbf{N}}$.

Example 2.7.3. Recall the homogeneous networks \mathbf{A} , \mathbf{B} and \mathbf{C} of Figure 2.1 and their input maps given in Example 3.4.1. In network \mathbf{A} , $\sigma_2^2 = \sigma_3^2 = \sigma_3 \circ \sigma_2 = \sigma_2 \circ \sigma_3 = \sigma_3$. Hence

$$\Sigma_{\mathbf{A}} = \{\sigma_1, \sigma_2, \sigma_3\}$$

is already a semigroup. In network \mathbf{B} , on the other hand, $\sigma_2^2 \neq \sigma_{1,2,3}$, so the collection $\{\sigma_1, \sigma_2, \sigma_3\}$ needs to be extended to obtain a collection

$$\Sigma_{\mathbf{B}} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} \text{ with } \sigma_4 = \sigma_2^2$$

that is closed under composition. Similarly, the input maps of network \mathbf{C} require an extension (in fact by two elements) to

$$\Sigma_{\mathbf{C}} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\} \text{ with } \sigma_4 = \sigma_2^2 \text{ and } \sigma_5 = \sigma_2 \circ \sigma_3.$$

The resulting input maps are the following.

A	v_1	v_2	v_3	B	v_1	v_2	v_3	C	v_1	v_2	v_3
σ_1	v_1	v_2	v_3	σ_1	v_1	v_2	v_3	σ_1	v_1	v_2	v_3
σ_2	v_1	v_1	v_2	σ_2	v_1	v_1	v_2	σ_2	v_2	v_2	v_1
σ_3	v_1	v_1	v_1	σ_3	v_2	v_2	v_2	σ_3	v_3	v_3	v_3
				σ_4	v_1	v_1	v_1	σ_4	v_2	v_2	v_2
								σ_5	v_1	v_1	v_1

One checks that the composition/product tables of $\Sigma_{\mathbf{A}}$, $\Sigma_{\mathbf{B}}$ and $\Sigma_{\mathbf{C}}$ read

$\Sigma_{\mathbf{A}}$	σ_1	σ_2	σ_3	$\Sigma_{\mathbf{B}}$	σ_1	σ_2	σ_3	σ_4
σ_1	σ_1	σ_2	σ_3	σ_1	σ_1	σ_2	σ_3	σ_4
σ_2	σ_2	σ_3	σ_3	σ_2	σ_2	σ_4	σ_4	σ_4
σ_3	σ_3	σ_3	σ_3	σ_3	σ_3	σ_3	σ_3	σ_3
				σ_4	σ_4	σ_4	σ_4	σ_4

$\Sigma_{\mathbf{C}}$	σ_1	σ_2	σ_3	σ_4	σ_5
σ_1	σ_1	σ_2	σ_3	σ_4	σ_5
σ_2	σ_2	σ_4	σ_5	σ_4	σ_4
σ_3	σ_3	σ_3	σ_3	σ_3	σ_3
σ_4	σ_4	σ_4	σ_4	σ_4	σ_4
σ_5	σ_5	σ_5	σ_5	σ_5	σ_5

One reads off the input maps $\tilde{\sigma}_1, \tilde{\sigma}_2$ and $\tilde{\sigma}_3$ of the lifts $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$. They are given by

$\tilde{\mathbf{A}}$	σ_1	σ_2	σ_3	$\tilde{\mathbf{B}}$	σ_1	σ_2	σ_3	σ_4
$\tilde{\sigma}_1$	σ_1	σ_2	σ_3	$\tilde{\sigma}_1$	σ_1	σ_2	σ_3	σ_4
$\tilde{\sigma}_2$	σ_2	σ_3	σ_3	$\tilde{\sigma}_2$	σ_2	σ_4	σ_4	σ_4
$\tilde{\sigma}_3$	σ_3	σ_3	σ_3	$\tilde{\sigma}_3$	σ_3	σ_3	σ_3	σ_3

$\tilde{\mathbf{C}}$	σ_1	σ_2	σ_3	σ_4	σ_5
$\tilde{\sigma}_1$	σ_1	σ_2	σ_3	σ_4	σ_5
$\tilde{\sigma}_2$	σ_2	σ_4	σ_5	σ_4	σ_4
$\tilde{\sigma}_3$	σ_3	σ_3	σ_3	σ_3	σ_3

The graphs of the fundamental networks $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ are depicted in Figure 2.3. The figure also displays the differential equations $\dot{X} = \gamma_{\tilde{f}}^{\tilde{\mathbf{A}}}(X)$, $\dot{X} = \gamma_{\tilde{f}}^{\tilde{\mathbf{B}}}(X)$ and $\dot{X} = \gamma_{\tilde{f}}^{\tilde{\mathbf{C}}}(X)$.

We note that network $\tilde{\mathbf{A}}$ is isomorphic to network \mathbf{A} . One may also observe that network $\tilde{\mathbf{B}}$ is isomorphic to a quotient of network $\tilde{\mathbf{B}}$ and that network $\tilde{\mathbf{C}}$ is isomorphic to a quotient of network $\tilde{\mathbf{C}}$. We show below that this is not a coincidence.

The following result clarifies the relation between a homogeneous network and its fundamental network.

Theorem 2.7.4. *Every homogeneous network $\mathbf{N} = \{A \rightrightarrows_t^s V\}$ is a quotient of its fundamental network $\tilde{\mathbf{N}}$. More precisely, for every vertex $v \in V$ of \mathbf{N} ,*

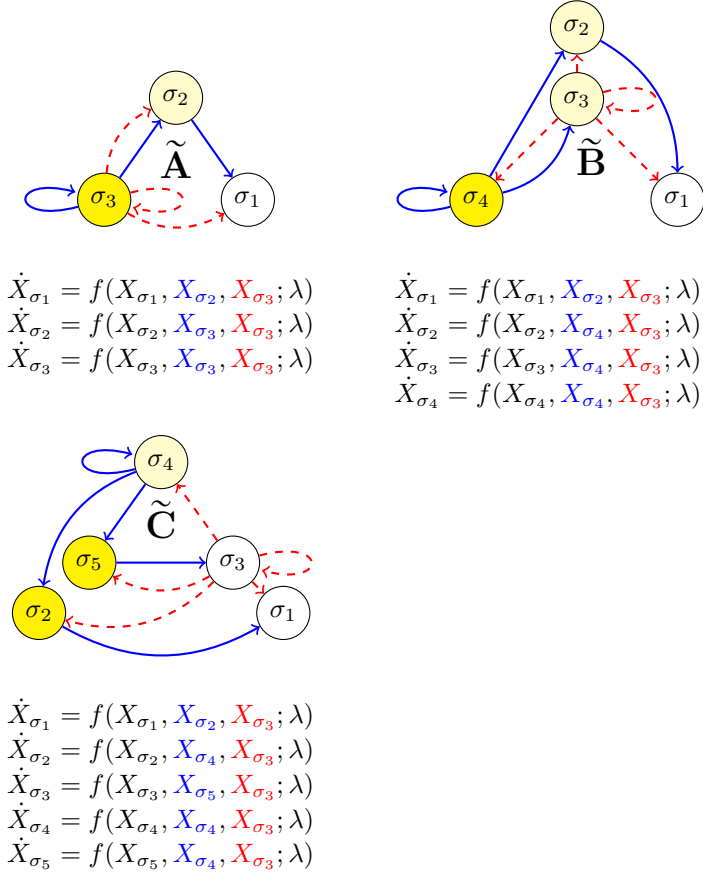


Figure 2.3: The fundamental networks of **A**, **B** and **C** and their equations of motion.

the map of vertices

$$\phi_v : \Sigma_{\mathbf{N}} \rightarrow V \text{ defined by } \phi_v(\sigma_j) := \sigma_j(v)$$

extends to a graph fibration from $\tilde{\mathbf{N}}$ to \mathbf{N} . In particular, the map $\phi_v^* : E_{\mathbf{N}} \rightarrow E_{\tilde{\mathbf{N}}}$ defined by $(\phi_v^* x)_{\sigma_j}(t) := x_{\phi_v(\sigma_j)} = x_{\sigma_j(v)}$ conjugates the network maps $\gamma_f^{\mathbf{N}}$ and $\gamma_f^{\tilde{\mathbf{N}}}$, that is

$$\phi_v^* \circ \gamma_f^{\mathbf{N}} = \gamma_f^{\tilde{\mathbf{N}}} \circ \phi_v^*.$$

Proof. It follows from the definition of $\tilde{\sigma}_k$ that

$$\sigma_k(\phi_v(\sigma_j)) = \sigma_k(\sigma_j(v)) = (\sigma_k \circ \sigma_j)(v) = (\tilde{\sigma}_k(\sigma_j))(v) = \phi_v(\tilde{\sigma}_k(\sigma_j)).$$

This shows that $\sigma_k \circ \phi_v = \phi_v \circ \tilde{\sigma}_k$ and thus by Proposition 2.6.3 that ϕ_v extends to a graph fibration. The remaining statements follow from Theorem 2.5.3. \square

The image of the map ϕ_v of Theorem 2.7.4 is equal to the subset $\{\sigma_j(v) \mid \sigma_j \in \Sigma_{\mathbf{N}}\}$ of the vertices of \mathbf{N} . Recall from Remark 2.7.1 that this set consists of all the direct and indirect inputs of cell v . We will introduce a special notation for these.

Definition 2.7.5. We define the *input network*

$$\mathbf{N}_{(v)} := \{A_{(v)} \rightrightarrows_t^s V_{(v)}\}$$

of a cell v in an arbitrary (i.e. not necessarily homogeneous) network $\mathbf{N} = \{A \rightrightarrows_t^s V\}$ by

$$V_{(v)} := \{w \in V \mid \exists \text{ path in } \mathbf{N} \text{ from } w \text{ to } v\} \text{ and } A_{(v)} := \{a \in A \mid t(a) \in V_{(v)}\}.$$

The input network $\mathbf{N}_{(v)}$ consists of those cells that can be “felt” by cell v , either directly or indirectly. In fact, it automatically holds that $s(a) \in V_{(v)}$ for all arrows $a \in A_{(v)}$. Hence, $\mathbf{N}_{(v)}$ is a subnetwork of \mathbf{N} and the embedding

$$e_{\mathbf{N}_{(v)}} : \mathbf{N}_{(v)} \rightarrow \mathbf{N}$$

is an injective graph fibration. Theorem 2.7.4 can now be rephrased as follows:

Corollary 2.7.6. *The dynamics of the input network $\mathbf{N}_{(v)}$ of every cell v of \mathbf{N} is embedded as the robust synchrony space*

$$\text{Syn}_{P_{(v)}} := \{X \in E_{\tilde{\mathbf{N}}} \mid X_{\sigma_j} = X_{\sigma_k} \text{ when } \sigma_j(v) = \sigma_k(v)\}$$

inside the dynamics of the fundamental network $\tilde{\mathbf{N}}$.

Proof. Theorem 2.7.4 implies that $\phi_v : \tilde{\mathbf{N}} \rightarrow \mathbf{N}_{(v)}$ is a surjective graph fibration for every cell v of \mathbf{N} . By Proposition 2.5.6, the linear map

$$\phi_v^* : E_{\mathbf{N}_{(v)}} \rightarrow E_{\tilde{\mathbf{N}}} \text{ defined by } (\phi_v^* x)_{\sigma_j} = x_{\sigma_j(v)}$$

is therefore injective. By Theorem 2.7.4, it thus embeds the dynamics of $\gamma_f^{\mathbf{N}_{(v)}}$ inside the dynamics of $\gamma_f^{\tilde{\mathbf{N}}}$. It is clear that $\text{im } \phi_v^* = \text{Syn}_{P_{(v)}}.$

One readily checks that the partition $P_{(v)}$ of $\Sigma_{\mathbf{N}}$ for which

$$\sigma_j \sim_{P_{(v)}} \sigma_k \text{ if and only if } \sigma_j(v) = \sigma_k(v)$$

is balanced. Indeed, if $\sigma_j \sim_{P(v)} \sigma_k$, then we have for every input map $\tilde{\sigma}_l$ of $\tilde{\mathbf{N}}$ that $(\tilde{\sigma}_l(\sigma_j))(v) = (\sigma_l \circ \sigma_j)(v) = \sigma_l(\sigma_j(v)) = \sigma_l(\sigma_k(v)) = (\sigma_l \circ \sigma_k)(v) = (\tilde{\sigma}_l(\sigma_k))(v)$, and hence that $\tilde{\sigma}_l(\sigma_j) \sim_{P(v)} \tilde{\sigma}_l(\sigma_k)$. Thus $P(v)$ is balanced.

Alternatively, one may recall from Theorem 2.7.4 that $\gamma_f^{\tilde{\mathbf{N}}} \circ \phi_v^* = \phi_v^* \circ \gamma_f^{\mathbf{N}}$ for any response function f . This implies in particular that

$$\gamma_f^{\tilde{\mathbf{N}}}(\text{im } \phi_v^*) \subset \text{im } \phi_v^*$$

and hence that $\text{im } \phi_v^*$ is invariant under the dynamics of $\gamma_f^{\tilde{\mathbf{N}}}$. \square

Remark 2.7.7. Identifying $E_{\mathbf{N}(v)}$ with the synchrony space $\text{Syn}_{P(v)} \subset E_{\tilde{\mathbf{N}}}$ by means of the embedding ϕ_v^* , we can also write the identity $\phi_v^* \circ \gamma_f^{\mathbf{N}} = \gamma_f^{\tilde{\mathbf{N}}} \circ \phi_v^*$ as

$$\gamma_f^{\mathbf{N}(v)} = \gamma_f^{\tilde{\mathbf{N}}} |_{E_{\mathbf{N}(v)}}.$$

In other words, we may think of the dynamics of $\mathbf{N}(v)$ as the restriction to a synchrony subspace of the dynamics of $\tilde{\mathbf{N}}$.

Remark 2.7.8. The dynamics of \mathbf{N} is itself embedded in the dynamics of $\tilde{\mathbf{N}}$ if there is a cell v in \mathbf{N} so that $\mathbf{N}(v) = \mathbf{N}$. It is natural to assume that such a cell exists: otherwise, the network may be considered pathological, or at least quite irrelevant for our understanding of network dynamics.

Remark 2.7.9. Theorem 2.7.4 shows that for every cell v in the homogeneous network \mathbf{N} , there is a graph fibration $\phi_v : \tilde{\mathbf{N}} \rightarrow \mathbf{N}$ that sends cell σ_1 of $\tilde{\mathbf{N}}$ (representing the unit of $\Sigma_{\mathbf{N}}$) to cell v of network \mathbf{N} . On the other hand, there is only one graph fibration $\phi : \tilde{\mathbf{N}} \rightarrow \mathbf{N}$ that maps cell σ_1 of $\tilde{\mathbf{N}}$ to cell v of \mathbf{N} , because if $\phi(\sigma_1) = v$, then $\phi(\sigma_k) = \phi(\tilde{\sigma}_k(\sigma_1)) = \sigma_k(\phi(\sigma_1)) = \sigma_k(v)$. So Theorem 2.7.4 in fact describes all possible graph fibrations from $\tilde{\mathbf{N}}$ to \mathbf{N} .

Example 2.7.10. Our networks \mathbf{A} , \mathbf{B} and \mathbf{C} are themselves input networks of one or more of their cells. For example, $\mathbf{A} = \mathbf{A}_{(v_3)}$, $\mathbf{B} = \mathbf{B}_{(v_3)}$ and $\mathbf{C} = \mathbf{C}_{(v_3)}$, so the networks are quotients of their fundamental networks. The corresponding graph fibrations are shown in Figure 2.4. For example, the graph fibration $\phi_{v_3} : \tilde{\mathbf{C}} \rightarrow \mathbf{C}$ sends

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \mapsto (v_3, v_1, v_3, v_2, v_1).$$

This means that when $(x_{v_1}(t), x_{v_2}(t), x_{v_3}(t))$ solves the equations of network \mathbf{C} , then

$$(X_{\sigma_1}(t), X_{\sigma_2}(t), X_{\sigma_3}(t), X_{\sigma_4}(t), X_{\sigma_5}(t)) = (x_{v_3}(t), x_{v_1}(t), x_{v_3}(t), x_{v_2}(t), x_{v_1}(t))$$

solves those of network $\tilde{\mathbf{C}}$. Network \mathbf{C} is therefore embedded inside network $\tilde{\mathbf{C}}$ as the robust synchrony space $\{X_{\sigma_1} = X_{\sigma_3}, X_{\sigma_2} = X_{\sigma_5}\}$. Similarly, network \mathbf{B} is realised inside $\tilde{\mathbf{B}}$ as the robust synchrony space $\{X_{\sigma_2} = X_{\sigma_3}\}$. Finally, the dynamics of networks \mathbf{A} and $\tilde{\mathbf{A}}$ are bi-conjugate because $\phi_{v_3} : \tilde{\mathbf{A}} \rightarrow \mathbf{A}$ is an isomorphism.

2.8 Hidden Symmetry

In this section, we prove the main results of this paper. We start with an observation.

Lemma 2.8.1. *The fundamental network $\tilde{\tilde{\mathbf{N}}}$ of a fundamental network $\tilde{\mathbf{N}}$ is isomorphic to $\tilde{\mathbf{N}}$.*

Proof. Recall that the vertex set of $\tilde{\mathbf{N}}$ is the semigroup $\Sigma_{\mathbf{N}} = \{\sigma_1, \dots, \sigma_n\}$, and that $\tilde{\mathbf{N}}$ has input maps $\tilde{\sigma}_1, \dots, \tilde{\sigma}_m$ defined by $\tilde{\sigma}_j(\sigma_k) = \sigma_j \circ \sigma_k$. Consequently, the vertex set of $\tilde{\tilde{\mathbf{N}}}$ is the semigroup $\Sigma_{\tilde{\mathbf{N}}}$ generated by $\tilde{\sigma}_1, \dots, \tilde{\sigma}_m$, while $\tilde{\tilde{\mathbf{N}}}$ has input maps $\tilde{\tilde{\sigma}}_1, \dots, \tilde{\tilde{\sigma}}_m$ defined by $\tilde{\tilde{\sigma}}_j(\tilde{\sigma}_k) = \tilde{\sigma}_j \circ \tilde{\sigma}_k$. We claim that $\Sigma_{\mathbf{N}}$ and $\Sigma_{\tilde{\mathbf{N}}}$ are isomorphic semigroups, which implies the lemma. To prove our claim, simply note that $(\tilde{\sigma}_k \circ \tilde{\sigma}_j)(\sigma_i) = \sigma_k \circ \sigma_j \circ \sigma_i = (\widetilde{\sigma_k \circ \sigma_j})(\sigma_i)$, i.e.

$$\tilde{\sigma}_k \circ \tilde{\sigma}_j = \widetilde{\sigma_k \circ \sigma_j}.$$

Because $\Sigma_{\mathbf{N}}$ is the smallest semigroup containing $\sigma_1, \dots, \sigma_m$, this observation implies that

$$\phi : \sigma_j \mapsto \tilde{\sigma}_j$$

defines a surjective homomorphism from $\Sigma_{\mathbf{N}}$ to $\Sigma_{\tilde{\mathbf{N}}}$. Moreover, ϕ is injective, because $\Sigma_{\mathbf{N}}$ contains a unit σ_1 , so that if $\tilde{\sigma}_j = \tilde{\sigma}_k$, then $\sigma_j = \sigma_j \circ \sigma_1 = \tilde{\sigma}_j(\sigma_1) = \tilde{\sigma}_k(\sigma_1) = \sigma_k \circ \sigma_1 = \sigma_k$. We conclude that ϕ is an isomorphism and, in particular, a bijection between the vertices of $\tilde{\mathbf{N}}$ and those of $\tilde{\tilde{\mathbf{N}}}$. It is also clear that ϕ intertwines the input maps of $\tilde{\mathbf{N}}$ and $\tilde{\tilde{\mathbf{N}}}$, since

$$\tilde{\tilde{\sigma}}_k(\phi(\sigma_j)) = \tilde{\tilde{\sigma}}_k(\tilde{\sigma}_j) = \tilde{\sigma}_k \circ \tilde{\sigma}_j = \widetilde{\sigma_k \circ \sigma_j} = \phi(\tilde{\sigma}_k(\sigma_j)).$$

This proves that ϕ extends to an isomorphism between $\tilde{\mathbf{N}}$ and $\tilde{\tilde{\mathbf{N}}}$. □

Combining Theorem 2.7.4 and Lemma 2.8.1, we obtain:

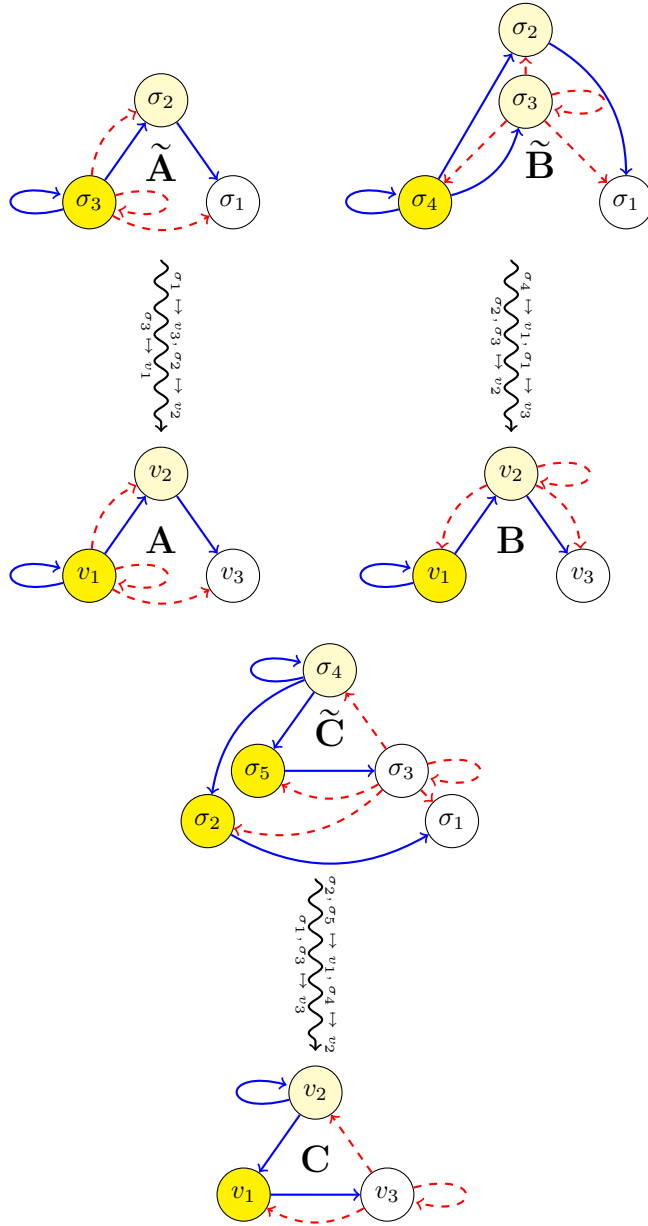


Figure 2.4: The graph fibrations $\phi_{v_3} : \tilde{\mathbf{A}} \rightarrow \mathbf{A}$ and $\phi_{v_3} : \tilde{\mathbf{B}} \rightarrow \mathbf{B}$ and $\phi_{v_3} : \tilde{\mathbf{C}} \rightarrow \mathbf{C}$.

Theorem 2.8.2. *Let $\tilde{\mathbf{N}}$ be a homogeneous fundamental network. For all $1 \leq i \leq n$, the map*

$$\phi_{\sigma_i} : \Sigma_{\mathbf{N}} \rightarrow \Sigma_{\mathbf{N}} \text{ defined by } \phi_{\sigma_i}(\sigma_j) := \sigma_j \circ \sigma_i$$

extends to a graph fibration from $\tilde{\mathbf{N}}$ to itself. Every network map $\gamma_f^{\tilde{\mathbf{N}}}$ is thus $\Sigma_{\mathbf{N}}$ -equivariant:

$$\phi_{\sigma_i}^* \circ \gamma_f^{\tilde{\mathbf{N}}} = \gamma_f^{\tilde{\mathbf{N}}} \circ \phi_{\sigma_i}^* \text{ for all } \sigma_i \in \Sigma_{\mathbf{N}},$$

where we recall that $\phi_{\sigma_i}^ : E_{\tilde{\mathbf{N}}} \rightarrow E_{\tilde{\mathbf{N}}}$ is defined by $(\phi_{\sigma_i}^* X)_{\sigma_j} := X_{\phi_{\sigma_i}(\sigma_j)} = X_{\sigma_j \circ \sigma_i}$.*

Proof. The statement of this theorem is a special case of the statement of Theorem 2.7.4, with \mathbf{N} replaced by $\tilde{\mathbf{N}}$ and $\tilde{\mathbf{N}}$ replaced by $\tilde{\mathbf{N}}$, noting that the latter is isomorphic to $\tilde{\mathbf{N}}$.

Alternatively, from the fact that left-multiplication and right-multiplication in $\Sigma_{\mathbf{N}}$ commute, it also follows directly that every ϕ_{σ_i} commutes with every input map $\tilde{\sigma}_k$ of $\tilde{\mathbf{N}}$:

$$\phi_{\sigma_i}(\tilde{\sigma}_k(\sigma_j)) = \sigma_k \circ \sigma_j \circ \sigma_i = \tilde{\sigma}_k(\phi_{\sigma_i}(\sigma_j)).$$

By Proposition 2.6.3 it thus follows that ϕ_{σ_i} extends to a graph fibration. The remaining statements of the theorem now follow from Theorem 2.7.4. \square

Remark 2.8.3. The maps $\phi_{\sigma_i} : \tilde{\mathbf{N}} \rightarrow \tilde{\mathbf{N}}$ of Theorem 2.8.2 are examples of graph fibrations from a network graph to itself. We shall refer to such graph fibrations as *self-fibrations*. The self-fibrations of a network need not form a group. For example, because the right-multiplication by σ_i in $\Sigma_{\mathbf{N}}$ need not be an invertible operation, the self-fibrations ϕ_{σ_i} need not be invertible. Nevertheless, because the composition of two graph fibrations is obviously a graph fibration, the self-fibrations of a network form a semigroup with unit.

Remark 2.8.4. Recall from Remark 2.7.9 that Theorem 2.8.2 describes all the self-fibrations of a fundamental network. It clearly holds that $(\phi_{\sigma_k} \circ \phi_{\sigma_j})(\sigma_i) = \sigma_i \circ \sigma_j \circ \sigma_k = \phi_{\sigma_j \circ \sigma_k}(\sigma_i)$, i.e.

$$\phi_{\sigma_k} \circ \phi_{\sigma_j} = \phi_{\sigma_j \circ \sigma_k}.$$

This contravariant transformation formula shows that the self-fibrations of a fundamental network $\tilde{\mathbf{N}}$ form a semigroup that is isomorphic to $\Sigma_{\mathbf{N}}^*$, the so-called *opposite semigroup* of $\Sigma_{\mathbf{N}}$, with product $\sigma_j * \sigma_k := \sigma_k \circ \sigma_j$.

Remark 2.8.5. On the other hand, Remark 2.5.5 implies the covariant transformation formula

$$\phi_{\sigma_j}^* \circ \phi_{\sigma_k}^* = (\phi_{\sigma_k} \circ \phi_{\sigma_j})^* = \phi_{\sigma_j \circ \sigma_k}^*$$

In particular, the assignment

$$\sigma_j \mapsto \phi_{\sigma_j}^* \text{ from } \Sigma_{\mathbf{N}} \text{ to } \mathfrak{gl}(E_{\tilde{\mathbf{N}}})$$

defines a representation of the semigroup $\Sigma_{\mathbf{N}}$ in the phase space of the fundamental network. This justifies that in Theorem 2.8.2 the fundamental network maps $\gamma_f^{\tilde{\mathbf{N}}}$ are called “ $\Sigma_{\mathbf{N}}$ -equivariant”. One could also say that $\Sigma_{\mathbf{N}}$ is a “symmetry-semigroup” of the fundamental network maps.

Remark 2.8.6. By Corollary 2.7.6, the dynamics of every input network $\mathbf{N}_{(v)}$ is embedded as the robust synchrony space $\text{Syn}_{P_{(v)}}$ inside the phase space of the fundamental network $\tilde{\mathbf{N}}$. Nevertheless, this synchrony space may not be invariant under the action of $\Sigma_{\mathbf{N}}$ on the phase space of the fundamental network, i.e. it may not hold that $\phi_{\sigma_j}^*(\text{Syn}_{P_{(v)}}) \subset \text{Syn}_{P_{(v)}}$ for all $\sigma_j \in \Sigma_{\mathbf{N}}$. Alternatively, if it so happens that $\phi_{\sigma_j}^*(\text{Syn}_{P_{(v)}}) \subset \text{Syn}_{P_{(v)}}$, then it is possible that $\phi_{\sigma_j}^*$ acts trivially on $\text{Syn}_{P_{(v)}}$ (i.e. fixes it pointwise).

All this means that $\Sigma_{\mathbf{N}}$ may not act (or not act faithfully) on the phase space of the network \mathbf{N} , but only on the extended phase space of its fundamental lift $\tilde{\mathbf{N}}$, in which that of \mathbf{N} is embedded. We think of the elements of $\Sigma_{\mathbf{N}}$ as *hidden symmetries* of \mathbf{N} . Perhaps counterintuitively, these hidden symmetries may have a major impact on the dynamics of \mathbf{N} , see for example Remark 2.8.10.

Example 2.8.7. Recall the fundamental networks $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ of Example 2.7.3 and Figure 2.3. Their self-fibrations can be read off from the product tables of $\Sigma_{\mathbf{A}}$, $\Sigma_{\mathbf{B}}$ and $\Sigma_{\mathbf{C}}$ given in Example 2.7.3. The action of these self-

fibrations on vertices is as follows:

$$\begin{array}{c|ccc} \tilde{\mathbf{A}} & \sigma_1 & \sigma_2 & \sigma_3 \\ \hline \phi_{\sigma_1} & \sigma_1 & \sigma_2 & \sigma_3 \\ \phi_{\sigma_2} & \sigma_2 & \sigma_3 & \sigma_3 \\ \phi_{\sigma_3} & \sigma_3 & \sigma_3 & \sigma_3 \end{array} \quad \begin{array}{c|cccc} \tilde{\mathbf{B}} & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \hline \phi_{\sigma_1} & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 \\ \phi_{\sigma_2} & \sigma_2 & \sigma_4 & \sigma_3 & \sigma_4 \\ \phi_{\sigma_3} & \sigma_3 & \sigma_4 & \sigma_3 & \sigma_4 \\ \phi_{\sigma_4} & \sigma_4 & \sigma_4 & \sigma_3 & \sigma_4 \end{array}$$

$$\begin{array}{c|ccccc} \tilde{\mathbf{C}} & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \\ \hline \phi_{\sigma_1} & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 \\ \phi_{\sigma_2} & \sigma_2 & \sigma_4 & \sigma_3 & \sigma_4 & \sigma_5 \\ \phi_{\sigma_3} & \sigma_3 & \sigma_5 & \sigma_3 & \sigma_4 & \sigma_5 \\ \phi_{\sigma_4} & \sigma_4 & \sigma_4 & \sigma_3 & \sigma_4 & \sigma_5 \\ \phi_{\sigma_5} & \sigma_5 & \sigma_4 & \sigma_3 & \sigma_4 & \sigma_5 \end{array} . \tag{2.8.1}$$

We note that, other than the identity ϕ_{σ_1} , none of these self-fibrations is invertible.

The symmetries of the equations of motion of the fundamental networks can in turn be read off from these tables. They are given by:

Network $\tilde{\mathbf{A}}$

$$\begin{aligned} \phi_{\sigma_1}^*(X) &= (X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}) \\ \phi_{\sigma_2}^*(X) &= (X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_3}) \\ \phi_{\sigma_3}^*(X) &= (X_{\sigma_3}, X_{\sigma_3}, X_{\sigma_3}) \end{aligned}$$

Network $\tilde{\mathbf{B}}$

$$\begin{aligned} \phi_{\sigma_1}^*(X) &= (X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}) \\ \phi_{\sigma_2}^*(X) &= (X_{\sigma_2}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}) \\ \phi_{\sigma_3}^*(X) &= (X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}) \\ \phi_{\sigma_4}^*(X) &= (X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}) \end{aligned}$$

Network $\tilde{\mathbf{C}}$

$$\begin{aligned} \phi_{\sigma_1}^*(X) &= (X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}) \\ \phi_{\sigma_2}^*(X) &= (X_{\sigma_2}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}) \\ \phi_{\sigma_3}^*(X) &= (X_{\sigma_3}, X_{\sigma_5}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}) \\ \phi_{\sigma_4}^*(X) &= (X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}) \\ \phi_{\sigma_5}^*(X) &= (X_{\sigma_5}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}, X_{\sigma_5}) \end{aligned} .$$

One may also check directly from the equations of motion that these maps send solutions to solutions. We remark that in network $\tilde{\mathbf{C}}$ the synchrony space $\{X_{\sigma_1} = X_{\sigma_3}, X_{\sigma_2} = X_{\sigma_5}\}$ (which is isomorphic to network \mathbf{C}) is only invariant under the symmetries $\phi_{\sigma_1}^*$ and $\phi_{\sigma_3}^*$, which both act trivially on it. This confirms that network \mathbf{C} does not admit any nontrivial symmetries, while its fundamental lift does. Similarly, in network \mathbf{B} , the synchrony space $\{X_{\sigma_2} = X_{\sigma_3}\}$ (which is isomorphic to network \mathbf{B}) is only invariant under the trivial symmetry $\phi_{\sigma_1}^*$. Network \mathbf{A} , on the other hand, is isomorphic to network $\tilde{\mathbf{A}}$, and is hence symmetric itself: it admits the full symmetry semigroup $\Sigma_{\mathbf{A}}$.

The following result emphasises the geometric importance of the hidden symmetries of the fundamental network. It states that they determine its robust synchronies.

Theorem 2.8.8. *Let $P = \{P_1, \dots, P_r\}$ be a balanced partition of the cells $\Sigma_{\mathbf{N}}$ of a homogeneous fundamental network $\tilde{\mathbf{N}}$ and let $\gamma : E_{\tilde{\mathbf{N}}} \rightarrow E_{\tilde{\mathbf{N}}}$ be any $\Sigma_{\mathbf{N}}$ -equivariant map. Then*

$$\gamma(\text{Syn}_P) \subset \text{Syn}_P.$$

Proof. Assume that $\gamma : E_{\tilde{\mathbf{N}}} \rightarrow E_{\tilde{\mathbf{N}}}$ is $\Sigma_{\mathbf{N}}$ -equivariant, i.e. that $\gamma \circ \phi_{\sigma_i}^* = \phi_{\sigma_i}^* \circ \gamma$ for all $\sigma_i \in \Sigma_{\mathbf{N}}$. This implies that

$$\begin{aligned} \gamma_{\sigma_i}(X) &= \gamma_{\sigma_1 \circ \sigma_i}(X) = (\phi_{\sigma_i}^* \gamma)_{\sigma_1}(X) = (\gamma \circ \phi_{\sigma_i}^*)_{\sigma_1}(X) \\ &= \gamma_{\sigma_1}(X_{\sigma_1 \circ \sigma_i}, \dots, X_{\sigma_n \circ \sigma_i}) = \gamma_{\sigma_1}(X_{\tilde{\sigma}_1(\sigma_i)}, \dots, X_{\tilde{\sigma}_n(\sigma_i)}). \end{aligned}$$

In other words, γ is a homogeneous network vector field with response function γ_{σ_1} on the network with vertex set $\Sigma_{\mathbf{N}}$ and with input maps $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$. Note that this does not imply that γ is a network vector field for the fundamental network $\tilde{\mathbf{N}}$, for which the input maps are $\tilde{\sigma}_1, \dots, \tilde{\sigma}_m$ (recall that m may be strictly less than n in general).

Now recall from Remark 2.6.4 that P is a balanced partition if and only if for all $1 \leq j \leq m$ and $1 \leq k \leq r$ there is an $1 \leq l \leq r$ such that $\tilde{\sigma}_j(P_k) \subset P_l$ (that is if $\tilde{\sigma}_1, \dots, \tilde{\sigma}_m$ preserve the partition). But the $\tilde{\sigma}_j$ with $m+1 \leq j \leq n$ are all of the form $\tilde{\sigma}_j = \tilde{\sigma}_{j_1} \circ \dots \circ \tilde{\sigma}_{j_q}$ for $1 \leq j_1, \dots, j_q \leq m$. Hence all the $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$ preserve the partition and the partition is automatically balanced for the extended network with input maps $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$. In particular, Syn_P is invariant under γ . \square

Remark 2.8.9. Let $\phi : \mathbf{N} \rightarrow \mathbf{N}$ be a self-fibration of a network and let $\gamma : E_{\mathbf{N}} \rightarrow E_{\mathbf{N}}$ be an equivariant map, i.e. $\phi^* \circ \gamma = \gamma \circ \phi^*$. Then $\text{Fix } \phi^* := \{x \in E_{\mathbf{N}} \mid \phi^* x = x\}$ is an example of an invariant subspace for γ , because

$\phi^*(\gamma(x)) = \gamma(\phi^*(x)) = \gamma(x)$ if $\phi^*(x) = x$. This is how invertible network symmetries (those that form the symmetry group of the network) yield invariant subspaces in a network dynamical system.

But when ϕ is not invertible, then one can imagine many more invariant subspaces induced by symmetry. For example, the image $\text{im } \phi^*$ of ϕ^* and the inverse image $(\phi^*)^{-1}(W)$ of a γ -invariant subspace W are invariant under the dynamics of γ .

Remark 2.8.10. Recall from Remark 2.7.7 that we may think of the phase space $E_{\mathbf{N}_{(v)}}$ of the input network $\mathbf{N}_{(v)}$ as a robust synchrony space in the phase space $E_{\tilde{\mathbf{N}}}$ of the fundamental network $\tilde{\mathbf{N}}$. It holds that $\gamma_f^{\mathbf{N}_{(v)}} = \gamma_{\tilde{f}}|_{E_{\mathbf{N}_{(v)}}$ and hence every robust synchrony space $\text{Syn}_P \subset E_{\mathbf{N}_{(v)}}$ for the dynamics of $\mathbf{N}_{(v)}$ is also a robust synchrony space for the dynamics of $\tilde{\mathbf{N}}$.

Theorem 2.8.8 states that not only the class of network maps $\gamma_{\tilde{f}}^{\tilde{\mathbf{N}}} : E_{\tilde{\mathbf{N}}} \rightarrow E_{\tilde{\mathbf{N}}}$ leaves $E_{\mathbf{N}_{(v)}}$ and $\text{Syn}_P \subset E_{\mathbf{N}_{(v)}}$ invariant, but the possibly much larger class of $\Sigma_{\mathbf{N}}$ -equivariant maps $\gamma : E_{\tilde{\mathbf{N}}} \rightarrow E_{\tilde{\mathbf{N}}}$ does so as well. Thus, one may argue here that robust synchrony is not “caused” by the network structure, but that it is a consequence of hidden symmetry.

2.9 Hidden Symmetry and Local Bifurcations

We saw that every homogeneous network is embedded in a network with semigroup symmetry, and it can be shown that a similar statement is true for nonhomogeneous networks. For networks without interchangeable inputs, hidden symmetry may even be thought of as “causing” robust synchrony. Similarly, it is conceivable that hidden symmetry generates many of the other intriguing phenomena that have been observed in networks, and that can not be explained from the existence of robust synchrony alone. This includes the surprising character of synchrony breaking bifurcations. In particular, it was shown in [23] that hidden symmetry can force spectral degeneracies at local singularities. As a result, seemingly anomalous bifurcations in networks may in fact be generic in classes of semigroup equivariant dynamical systems. This is for example the case for the synchrony breaking bifurcations in networks **A**, **B** and **C** discussed in Section 2.3.

In addition, (hidden) symmetry is easier to incorporate in the analysis of network systems than “network structure”, if only because hidden symmetry is not lost under coordinate changes and is therefore an intrinsic property of a dynamical system. The analysis of networks may thus become much simpler when hidden symmetries are taken into account. This is certainly true for dynamical systems with “classical” symmetries [6, 14, 15]. As a re-

sult, many generic phenomena in dynamical systems with compact symmetry groups have been classified, and there exists a well-developed theory of local bifurcations for dynamical systems with compact symmetry groups. This theory relies on representation theory, equivariant singularity theory, and (group-)equivariant counterparts of the most important methods from local bifurcation theory, such as *normal form reduction*, *Lyapunov-Schmidt reduction* and *centre manifold reduction*. Neither of these theories and methods admits a natural generalisation to systems with a network structure. On the other hand, it turns out that (hidden) semigroup symmetry can be preserved in all three aforementioned reduction methods. For normal form reduction this was essentially proved in [21], and for Lyapunov-Schmidt reduction in [23]. For centre manifold reduction the situation is more technical. How semigroup symmetry affects a centre manifold, is the topic of our paper [19].

It is not our goal to develop the local bifurcation theory of dynamical systems with semigroup symmetry any further in this paper. Instead, we shall briefly sketch now how hidden symmetry can impact local bifurcations, at the hand of our example networks **A**, **B** and **C**. We claim that the synchrony breaking bifurcations in these networks that were discussed in Section 2.3, are determined by hidden symmetry, and we will sketch how this can be proved. We stress that this section is only meant as an illustration of the importance of hidden symmetry for the synchrony breaking behaviour of networks. Several claims that are made in this section have been or will be proved elsewhere.

We start with recalling some general theory from [23]. First of all, when Σ is a semigroup and W a finite dimensional real vector space, then we call a map

$$A : \Sigma \rightarrow \mathfrak{gl}(W) \text{ for which } A_{\sigma_i} \circ A_{\sigma_j} = A_{\sigma_i \circ \sigma_j} \text{ for all } \sigma_i, \sigma_j \in \Sigma$$

a *representation* of the semigroup Σ in W . A subspace $W_1 \subset W$ is called a *subrepresentation* of W if $A_{\sigma_i}(W_1) \subset W_1$ for all $\sigma_i \in \Sigma$. The smallest subrepresentations that build up a given representation, have a special name:

Definition 2.9.1. A subrepresentation $W_1 \subset W$ of Σ is called *indecomposable* if W_1 is not a direct sum $W_1 = W_2 \oplus W_3$ with W_2 and W_3 both nonzero subrepresentations of W_1 .

Unlike so-called *irreducible* subrepresentations, indecomposable subrepresentations may contain nontrivial subrepresentations, but these can then not be complemented by another nontrivial subrepresentation. By definition, every representation is a direct sum of indecomposable subrepresentations. Moreover, by the Krull-Schmidt theorem [23], the decomposition

of a representation into indecomposable subrepresentations is unique up to isomorphism.

When $A : \Sigma \rightarrow \mathfrak{gl}(W)$ is a representation and $L : W \rightarrow W$ is a linear map so that

$$L \circ A_{\sigma_j} = A_{\sigma_j} \circ L \text{ for all } \sigma_j \in \Sigma,$$

then we call L an *endomorphism* of W and write $L \in \text{End}(W)$.

Remark 2.9.2. Recall from Theorem 2.8.2 that each fundamental network map $\gamma_f^{\tilde{\mathbf{N}}} : E_{\tilde{\mathbf{N}}} \rightarrow E_{\tilde{\mathbf{N}}}$ is $\Sigma_{\mathbf{N}}$ -equivariant, i.e. $\gamma_f^{\tilde{\mathbf{N}}} \circ \phi_{\sigma_i}^* = \phi_{\sigma_i}^* \circ \gamma_f^{\tilde{\mathbf{N}}}$ for all $\sigma_i \in \Sigma_{\mathbf{N}}$. Differentiation of this identity at a fully synchronous (and hence fixed by $\Sigma_{\mathbf{N}}$) point (say $X = 0$) yields that

$$L \circ \phi_{\sigma_i}^* = \phi_{\sigma_i}^* \circ L \text{ for } L := D_X \gamma_f^{\tilde{\mathbf{N}}}(0).$$

In other words, the linearisation of a fundamental network map at a fully synchronous point is an example of an endomorphism of the representation of $\Sigma_{\mathbf{N}}$ in $E_{\tilde{\mathbf{N}}}$.

When $\lambda \in \mathbb{R}$ is an eigenvalue of an endomorphism $L \in \text{End}(W)$, the generalized eigenspace

$$\mathbb{E}_\lambda := \ker(L - \lambda \text{Id}_W)^{\dim W}$$

is a subrepresentation of W , and the same is true for the real generalized eigenspaces of the complex eigenvalues of L . It follows that the (unique) splitting of W into indecomposable subrepresentations determines to a large extent the spectral properties of its endomorphisms, and this explains how symmetry and hidden symmetry can force the linearisation matrix of a network map to have a degenerate spectrum. See [23] for more precise statements on the relation between indecomposable subrepresentations and the spectrum of endomorphisms.

Example 2.9.3. Recall the fundamental network maps $\gamma_f^{\tilde{\mathbf{A}}}$, $\gamma_f^{\tilde{\mathbf{B}}}$ and $\gamma_f^{\tilde{\mathbf{C}}}$ given in Figure 2.3. Assume now that the cells in the networks are 1-dimensional (that is $X_{\sigma_i} \in \mathbb{R}$ for all σ_i). Then the linearisation $L_{\tilde{\mathbf{A}}} := D_X \gamma_f^{\tilde{\mathbf{A}}}(0; 0)$ has the form (writing $a := D_1 f(0; 0) \in \mathbb{R}$ etc.)

$$L_{\tilde{\mathbf{A}}} = \begin{pmatrix} a & b & c \\ 0 & a & b+c \\ 0 & 0 & a+b+c \end{pmatrix}.$$

When $b + c \neq 0$, then $L_{\tilde{\mathbf{A}}}$ has an eigenvalue $a + b + c$ with algebraic and geometric multiplicity 1 and an eigenvalue a with algebraic multiplicity 2

and geometric multiplicity 1. The generalized eigenspaces of $L_{\tilde{\mathbf{A}}}$ are

$$\begin{aligned}\mathbb{E}_{a+b+c} &= \{X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_3}\} \text{ and} \\ \mathbb{E}_a &= \{X_{\sigma_3} = 0\}.\end{aligned}$$

Recall that $L_{\tilde{\mathbf{A}}}$ is an endomorphism of the representation of $\Sigma_{\mathbf{A}}$ (this representation was given in Example 2.8.7). It turns out that \mathbb{E}_{a+b+c} and \mathbb{E}_a both are indecomposable subrepresentations of $\Sigma_{\mathbf{A}}$. Because the splitting of a representation into indecomposable summands is unique up to isomorphism, it follows that every endomorphism of $\Sigma_{\mathbf{A}}$ can have at most 2 generalized eigenspaces. Thus, the spectral degeneracy of $L_{\tilde{\mathbf{A}}}$ is a consequence of symmetry. Because networks \mathbf{A} and $\tilde{\mathbf{A}}$ are isomorphic, the double degeneracy of the eigenvalue a in network \mathbf{A} is also a result of $\Sigma_{\mathbf{A}}$ -equivariance.

Similar considerations apply to $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$. The linearisation $L_{\tilde{\mathbf{B}}} := D_X \gamma_f^{\tilde{\mathbf{B}}}(0; 0)$ reads

$$L_{\tilde{\mathbf{B}}} = \begin{pmatrix} a & b & c & 0 \\ 0 & a & c & b \\ 0 & 0 & a+c & b \\ 0 & 0 & c & a+b \end{pmatrix}.$$

When $b+c \neq 0$, its generalized eigenspaces are

$$\begin{aligned}\mathbb{E}_{a+b+c} &= \{X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4}\} \text{ and} \\ \mathbb{E}_a &= \{cX_{\sigma_3} + bX_{\sigma_4} = 0\}.\end{aligned}$$

Both are indecomposable subrepresentations of $\Sigma_{\mathbf{B}}$. In addition, equivariance implies that $L_{\tilde{\mathbf{B}}}$ leaves the synchrony space $\{X_{\sigma_2} = X_{\sigma_3}\}$ (that is, network \mathbf{B}) invariant. This synchrony space intersects \mathbb{E}_a in a 2-dimensional subspace, and this explains the double degeneracy of the eigenvalue a in network \mathbf{B} . Finally, the linearisation matrix $L_{\tilde{\mathbf{C}}} := D_X \gamma_f^{\tilde{\mathbf{C}}}(0; 0)$ is

$$L_{\tilde{\mathbf{C}}} = \begin{pmatrix} a & b & c & 0 & 0 \\ 0 & a & c & b & 0 \\ 0 & 0 & a+c & 0 & b \\ 0 & 0 & c & a+b & 0 \\ 0 & 0 & c & b & a \end{pmatrix}.$$

When $b+c \neq 0$, it has generalized eigenspaces

$$\begin{aligned}\mathbb{E}_{a+b+c} &= \{X_{\sigma_1} = X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4} = X_{\sigma_5}\} \text{ and} \\ \mathbb{E}_a &= \{c(b+c)X_{\sigma_3} + b^2X_{\sigma_4} + bcX_{\sigma_5} = 0\}.\end{aligned}$$

The degenerate eigenvalue a now has algebraic multiplicity 4 and geometric multiplicity 1. Both generalized eigenspaces are indecomposable subrepresentations of $\Sigma_{\mathbf{C}}$. Moreover, \mathbb{E}_a intersects the robust synchrony space $\{X_{\sigma_1} = X_{\sigma_3}, X_{\sigma_2} = X_{\sigma_5}\}$ (that is, network \mathbf{C}) in a two-dimensional subspace. This explains the double degeneracy of the eigenvalue a in network \mathbf{C} .

Hidden symmetries do not only affect the linear, but also the nonlinear terms of network maps. One can therefore expect different nonlinear dynamics and bifurcations in networks with non-isomorphic (hidden) symmetry semigroups. Indeed, this is what explains the different character of the synchrony breaking bifurcations in networks \mathbf{A} , \mathbf{B} and \mathbf{C} .

These bifurcations can be investigated with various classical methods, including normal form reduction, centre manifold reduction and Lyapunov-Schmidt reduction. We will use the remainder of this section to sketch how Lyapunov-Schmidt reduction (which is perhaps the simplest of these methods) can predict the local asymptotics of the synchrony breaking steady state branches of a fundamental network $\tilde{\mathbf{N}}$. In principle, information about the stability of solution branches can not be obtained with this method.

So let us study the steady states of a parameter dependent fundamental network map

$$\gamma_f^{\tilde{\mathbf{N}}} : E_{\tilde{\mathbf{N}}} \times \Lambda \rightarrow E_{\tilde{\mathbf{N}}} \text{ with } \Lambda \subset \mathbb{R}^p \text{ an open set of parameters,}$$

near a given synchronous steady state (say $X = 0$) and given parameter value (say $\lambda = 0$). Thus, we assume that $\gamma_f^{\tilde{\mathbf{N}}}(0; 0) = 0$. Synchrony breaking can occur when the linearisation $L := D_X \gamma_f^{\tilde{\mathbf{N}}}(0; 0)$ is nonsynchronously degenerate, i.e. when

$$\mathbb{E}_0 := \text{gen ker } L \not\subset \{X_{\sigma_1} = \dots = X_{\sigma_n}\}.$$

Lyapunov-Schmidt reduction is a method to reduce the steady state equation $\gamma_f^{\tilde{\mathbf{N}}}(X; \lambda) = 0$, locally near $(X; \lambda) = (0; 0)$, to an equivalent equation of the form

$$F(X; \lambda) = 0 \text{ for } F : \mathbb{E}_0 \times \Lambda \rightarrow \mathbb{E}_0 \text{ defined near } (0; 0).$$

It was proved in [23] that it can be arranged that this F inherits $\Sigma_{\mathbf{N}}$ -equivariance from $\gamma_f^{\tilde{\mathbf{N}}}$ (recall that $\Sigma_{\mathbf{N}}$ restricts to a representation on \mathbb{E}_0). Equivariance now imposes restrictions on F that impact the solutions of the reduced bifurcation equation $F(X; \lambda) = 0$.

Moreover, if $\text{Syn}_P \subset E_{\tilde{\mathbf{N}}}$ is any robust synchrony space, then equivariance implies that

$$F(\mathbb{E}_0 \cap \text{Syn}_P; \lambda) \subset \mathbb{E}_0 \cap \text{Syn}_P,$$

even when $\mathbb{E}_0 \cap \text{Syn}_P$ is not a subrepresentation of $\Sigma_{\mathbf{N}}$. In this way, Lyapunov-Schmidt reduction replaces the problem of finding synchronous steady states of $\gamma_f^{\tilde{\mathbf{N}}}$ by the problem of finding zeroes of

$$F : \mathbb{E}_0 \cap \text{Syn}_P \times \Lambda \rightarrow \mathbb{E}_0 \cap \text{Syn}_P.$$

This may entail a considerable dimension reduction of the bifurcation problem.

We shall now illustrate how these observations can be used to predict the asymptotics of generic synchrony breaking steady state branches in networks $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$.

Example 2.9.4. Example 2.9.3 shows that network $\tilde{\mathbf{A}}$ can only break synchrony when $a = D_1 f(0; 0) = 0$. Assuming that $a = 0$ and $b + c \neq 0$, it holds that

$$\mathbb{E}_0 = \{X_{\sigma_3} = 0\}.$$

Let us coordinatise \mathbb{E}_0 with the variables $(X_{\sigma_1}, X_{\sigma_2})$, and accordingly write $F = (F_{\sigma_1}, F_{\sigma_2})$. In these coordinates, the action of $\Sigma_{\mathbf{A}}$ (see Example 2.8.7) on \mathbb{E}_0 is given by

$$\begin{aligned} \phi_{\sigma_1}^*(X_{\sigma_1}, X_{\sigma_2}) &= (X_{\sigma_1}, X_{\sigma_2}), \\ \phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}) &= (X_{\sigma_2}, 0), \\ \phi_{\sigma_3}^*(X_{\sigma_1}, X_{\sigma_2}) &= (0, 0). \end{aligned}$$

The equivariance of F under $\phi_{\sigma_2}^*$ now gives the identities

$$\begin{aligned} F_{\sigma_2}(X_{\sigma_1}, X_{\sigma_2}; \lambda) &= (\phi_{\sigma_2}^* F)_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}; \lambda) \\ &= F_{\sigma_1}(\phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}); \lambda) = F_{\sigma_1}(X_{\sigma_2}, 0; \lambda) \text{ and} \\ 0 &= (\phi_{\sigma_2}^* F)_{\sigma_2}(X_{\sigma_1}, X_{\sigma_2}; \lambda) = F_{\sigma_2}(\phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}); \lambda) \\ &= F_{\sigma_2}(X_{\sigma_2}, 0; \lambda). \end{aligned}$$

In other words, the map F is of the form

$$F(X_{\sigma_1}, X_{\sigma_2}; \lambda) = (F_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}; \lambda), F_{\sigma_1}(X_{\sigma_2}, 0; \lambda)) \text{ with } F_{\sigma_1}(0, 0; \lambda) = 0.$$

Also, every F that is of this form (for some smooth function F_{σ_1}) is $\Sigma_{\mathbf{A}}$ -equivariant. Because the bifurcation equation $F = 0$ has a special form, one may expect its local solutions to have a special structure as well. Indeed, when $\lambda \in \Lambda := \mathbb{R}$ and F_{σ_1} admits the generic expansion

$$\begin{aligned} F_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}; \lambda) &= \alpha \lambda X_{\sigma_1} + b X_{\sigma_2} + A X_1^2 \\ &\quad + \mathcal{O}(|X_{\sigma_1}| |\lambda|^2 + |X_{\sigma_1}|^3 + |X_{\sigma_2}| |X| + |X_{\sigma_2}| |\lambda|), \end{aligned} \quad (2.9.1)$$

then it follows that

$$F_{\sigma_2}(X_{\sigma_1}, X_{\sigma_2}; \lambda) = \alpha\lambda X_{\sigma_2} + AX_{\sigma_2}^2 + \mathcal{O}(|X_{\sigma_2}| \cdot |\lambda|^2 + |X_{\sigma_2}|^3).$$

Under the nondegeneracy conditions that $\alpha, \textcolor{blue}{b}, A \neq 0$, the equation $F_{\sigma_2} = 0$ yields that $X_{\sigma_2} = 0$ or $X_{\sigma_2} = -\frac{\alpha}{A}\lambda + \mathcal{O}(\lambda^2)$. In the first case, the equation $F_{\sigma_1} = 0$ gives that either $X_{\sigma_1} = 0$ or $X_{\sigma_1} = -\frac{\alpha}{A}\lambda + \mathcal{O}(\lambda^2)$. In the second case, we find that $X_{\sigma_1} = \pm\sqrt{\frac{b\alpha}{A^2}}\lambda + \mathcal{O}(\lambda)$. As a result, one can expect network $\tilde{\mathbf{A}}$ to generically support three solution branches near $(X; \lambda) = (0; 0)$. They have the asymptotics

$$\begin{aligned} X_{\sigma_1} &= X_{\sigma_2} = X_{\sigma_3} = 0, \\ X_{\sigma_1} &\sim \lambda, X_{\sigma_2} = X_{\sigma_3} = 0 \text{ and} \\ X_{\sigma_1} &\sim \pm\sqrt{\lambda}, X_{\sigma_2} \sim \lambda, X_{\sigma_3} = 0. \end{aligned}$$

These branches lie on one $\Sigma_{\mathbf{A}}$ -orbit. Moreover, recalling from Example 2.4 that $x_{v_1} = X_{\sigma_3}, x_{v_2} = X_{\sigma_2}, x_{v_3} = X_{\sigma_1}$, this shows that the bifurcation in network \mathbf{A} displayed in Table 2.1 is a generic equivariant bifurcation. A proof of this can also be found in [22, 23].

Example 2.9.5. Also network $\tilde{\mathbf{B}}$ can only break synchrony when $a = 0$. Assuming this and $\textcolor{blue}{b} + \textcolor{red}{c} \neq 0$, we recall from Example 2.9.3 that

$$\mathbb{E}_0 = \{\textcolor{red}{c}X_{\sigma_3} + \textcolor{blue}{b}X_{\sigma_4} = 0\}.$$

When $\textcolor{blue}{b} \neq 0$, we may coordinatise \mathbb{E}_0 by $(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3})$, letting $X_{\sigma_4} = -\frac{\textcolor{red}{c}}{\textcolor{blue}{b}}X_{\sigma_3}$. Similarly we coordinatise $F: \mathbb{E}_0 \times \mathbb{R} \rightarrow \mathbb{E}_0$ as $F = (F_{\sigma_1}, F_{\sigma_2}, F_{\sigma_3})$. The action of $\Sigma_{\mathbf{B}}$ on \mathbb{E}_0 is then

$$\begin{aligned} \phi_{\sigma_1}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}) &= (X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}), \\ \phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}) &= (X_{\sigma_2}, -\frac{\textcolor{red}{c}}{\textcolor{blue}{b}}X_{\sigma_3}, X_{\sigma_3}), \\ \phi_{\sigma_3}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}) &= (X_{\sigma_3}, -\frac{\textcolor{red}{c}}{\textcolor{blue}{b}}X_{\sigma_3}, X_{\sigma_3}), \\ \phi_{\sigma_4}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}) &= (-\frac{\textcolor{red}{c}}{\textcolor{blue}{b}}X_{\sigma_3}, -\frac{\textcolor{red}{c}}{\textcolor{blue}{b}}X_{\sigma_3}, X_{\sigma_3}). \end{aligned}$$

The equivariance of F implies among others that

$$\begin{aligned} F_{\sigma_2}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}; \lambda) &= (\phi_{\sigma_2}^* F)_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}; \lambda) \\ &= F_{\sigma_1}(\phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}); \lambda) \\ &= F_{\sigma_1}(X_{\sigma_2}, -\frac{c}{b}X_{\sigma_3}, X_{\sigma_3}; \lambda) \text{ and} \end{aligned} \quad (2.9.2)$$

$$\begin{aligned} F_{\sigma_3}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}; \lambda) &= (\phi_{\sigma_3}^* F)_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}; \lambda) \\ &= F_{\sigma_1}(\phi_{\sigma_3}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}); \lambda) \\ &= F_{\sigma_1}(X_{\sigma_3}, -\frac{c}{b}X_{\sigma_3}, X_{\sigma_3}; \lambda). \end{aligned} \quad (2.9.3)$$

In particular, it holds that $F_{\sigma_2} = F_{\sigma_3}$ if $X_{\sigma_2} = X_{\sigma_3}$ and we see that F leaves the robust synchrony space $\mathbb{E}_0 \cap \{X_{\sigma_2} = X_{\sigma_3}\}$ (that is, network **B**) invariant. Moreover, the remaining restrictions on $F_{\sigma_1}, F_{\sigma_2}, F_{\sigma_3}$ imposed by equivariance can be formulated as restrictions on F_{σ_1} . It turns out that they all reduce to a single additional restriction:

$$-\frac{c}{b}F_{\sigma_1}(X_{\sigma_3}, -\frac{c}{b}X_{\sigma_3}, X_{\sigma_3}; \lambda) = F_{\sigma_1}(-\frac{c}{b}X_{\sigma_3}, -\frac{c}{b}X_{\sigma_3}, X_{\sigma_3}; \lambda).$$

Zeroes of F inside $\{X_{\sigma_2} = X_{\sigma_3}\}$ thus correspond to zeroes of

$$G(X_{\sigma_1}, X_{\sigma_2}; \lambda) := \begin{pmatrix} F_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_2}; \lambda) \\ F_{\sigma_1}(X_{\sigma_2}, -\frac{c}{b}X_{\sigma_2}, X_{\sigma_2}; \lambda) \end{pmatrix},$$

with the above restriction on the otherwise arbitrary function F_{σ_1} . If we assume for instance that $\lambda \in \Lambda := \mathbb{R}$ and that F_{σ_1} admits the generic expansion

$$\begin{aligned} F_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}; \lambda) &= \alpha\lambda X_{\sigma_1} + (b + \beta\lambda)X_{\sigma_2} + (c + \gamma\lambda)X_{\sigma_3} \\ &\quad + AX_{\sigma_1}^2 + BX_{\sigma_1}X_{\sigma_2} + CX_{\sigma_2}^2 + DX_{\sigma_1}X_{\sigma_3} + EX_{\sigma_2}X_{\sigma_3} + FX_{\sigma_3}^2 \\ &\quad + \mathcal{O}(\|X\|^3 + |\lambda| \cdot \|X\|^2 + |\lambda|^2 \cdot \|X\|), \end{aligned} \quad (2.9.4)$$

then it follows from the condition on F_{σ_1} that $\beta c - \gamma b = Abc + Cc^2 - Ebc + Fb^2 = 0$. Also,

$$\begin{aligned} G(X; \lambda) &= \\ &\begin{pmatrix} \alpha\lambda X_{\sigma_1} + (b + c)X_{\sigma_2} + AX_{\sigma_1}^2 \\ \quad + \mathcal{O}(|\lambda| \cdot |X_{\sigma_2}| + |X_{\sigma_1}| \cdot |X_{\sigma_2}| + |X_{\sigma_2}|^2 + \|X\|^3 + |\lambda| \cdot \|X\|^2 + |\lambda|^2 \cdot \|X\|) \\ \alpha\lambda X_{\sigma_2} + HX_{\sigma_2}^2 + \mathcal{O}(|X_{\sigma_2}|^3 + |\lambda| \cdot |X_{\sigma_2}|^2 + |\lambda|^2 \cdot |X_{\sigma_2}|) \end{pmatrix} \end{aligned}$$

in which $H := A - \frac{c}{b}B + \frac{c^2}{b^2}C + D - \frac{c}{b}E + F$. Under the nondegeneracy conditions that $\alpha, A, H \neq 0$, the equation $F_{\sigma_2} = 0$ now gives that $X_{\sigma_2} = 0$

or $X_{\sigma_2} = -\frac{\alpha}{H}\lambda + \mathcal{O}(\lambda^2)$. In the first case, the equation $F_{\sigma_1} = 0$ gives that $X_{\sigma_1} = 0$ or $X_{\sigma_1} = -\frac{\alpha}{A}\lambda + \mathcal{O}(\lambda^2)$. In the second case we find that $X_{\sigma_1} = \pm\sqrt{\frac{\alpha(b+c)}{AH}}\lambda + \mathcal{O}(\lambda)$. Using our assumption that $X_{\sigma_2} = X_{\sigma_3}$ and the relation $X_{\sigma_4} = -\frac{b}{c}X_{\sigma_3}$, this yields three generic local steady state branches:

$$\begin{aligned} X_{\sigma_1} &= X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4} = 0, \\ X_{\sigma_1} &\sim \lambda, X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4} = 0 \text{ and} \\ X_{\sigma_1} &\sim \pm\sqrt{\lambda}, X_{\sigma_2} = X_{\sigma_3} \sim \lambda, X_{\sigma_4} = -\frac{c}{b}X_{\sigma_3} \sim \lambda. \end{aligned}$$

These branches are not related by symmetry. On the other hand, because $x_{v_1} = X_{\sigma_4}$, $x_{v_2} = X_{\sigma_2} = X_{\sigma_3}$, $x_{v_3} = X_{\sigma_1}$, we have proved that the steady state asymptotics of network **B** in Table 2.1 is generic in systems with hidden $\Sigma_{\mathbf{B}}$ -symmetry.

Example 2.9.6. As for the previous examples, network $\tilde{\mathbf{C}}$ can only break synchrony when $a = 0$. If we assume this and demand that $b + c \neq 0$, then it follows from Example 2.9.3 that

$$\mathbb{E}_0 = \{c(b+c)X_{\sigma_3} + b^2X_{\sigma_4} + bcX_{\sigma_5} = 0\}.$$

In the generic situation that $b, c \neq 0$, let us coordinatise this subspace by $(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4})$. In particular, we then have that $X_{\sigma_5} = -\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}$. Moreover, the action of $\Sigma_{\mathbf{C}}$ on \mathbb{E}_0 is given in these coordinates by

$$\begin{aligned} \phi_{\sigma_1}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}) &= (X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}), \\ \phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}) &= (X_{\sigma_2}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}), \\ \phi_{\sigma_3}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}) &= (X_{\sigma_3}, -\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}), \\ \phi_{\sigma_4}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}) &= (X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}), \\ \phi_{\sigma_5}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}) &= (-\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}). \end{aligned}$$

Coordinatising $F : \mathbb{E}_0 \times \Lambda \rightarrow \mathbb{E}_0$ as $F = (F_{\sigma_1}, F_{\sigma_2}, F_{\sigma_3}, F_{\sigma_4})$, we see that $\Sigma_{\mathbf{C}}$ -equivariance implies among others that the F_{σ_i} can be expressed in terms of F_{σ_1} . For example,

$$\begin{aligned} F_{\sigma_2}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}; \lambda) &= (\phi_{\sigma_2}^* F)_{\sigma_1}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}; \lambda) \\ &= F_{\sigma_1}(\phi_{\sigma_2}^*(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}); \lambda) \\ &= F_{\sigma_1}(X_{\sigma_2}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}; \lambda), \end{aligned}$$

and similarly,

$$\begin{aligned}
 F_{\sigma_3}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}; \lambda) &= \\
 F_{\sigma_1}(X_{\sigma_3}, -\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}; \lambda) \\
 F_{\sigma_4}(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}, X_{\sigma_4}; \lambda) &= F_{\sigma_1}(X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}; \lambda).
 \end{aligned} \tag{2.9.5}$$

It turns out that equivariance is met precisely when F_{σ_1} satisfies the additional condition

$$\begin{aligned}
 F_{\sigma_1}(-\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}; \lambda) \\
 = -\frac{b+c}{b}F_{\sigma_1}(X_{\sigma_3}, -\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}; \lambda) \\
 - \frac{b}{c}F_{\sigma_1}(X_{\sigma_4}, X_{\sigma_4}, X_{\sigma_3}, X_{\sigma_4}; \lambda).
 \end{aligned} \tag{2.9.6}$$

In particular, one may verify that $F_{\sigma_1} = F_{\sigma_3}$ and $F_{\sigma_2} = -\frac{b+c}{b}F_{\sigma_3} - \frac{b}{c}F_{\sigma_4}$ whenever $X_{\sigma_1} = X_{\sigma_3}$ and $X_{\sigma_2} = -\frac{b+c}{b}X_{\sigma_3} - \frac{b}{c}X_{\sigma_4}$. This confirms that F leaves the robust synchrony space $\mathbb{E}_0 \cap \{X_{\sigma_1} = X_{\sigma_3}, X_{\sigma_2} = X_{\sigma_5}\}$ invariant - recall that this corresponds to network **C**.

We will choose X_{σ_2} and X_{σ_4} as the free variables in this restricted system, and write

$$X_{\sigma_1} = X_{\sigma_3} = -\frac{b}{b+c}X_{\sigma_2} - \frac{b^2}{c(b+c)}X_{\sigma_4}.$$

It thus follows that we are searching for zeroes of the map

$$G(X_{\sigma_2}, X_{\sigma_4}; \lambda) := \begin{pmatrix} F_{\sigma_1}(X_{\sigma_2}, X_{\sigma_4}, -\frac{b}{b+c}X_{\sigma_2} - \frac{b^2}{c(b+c)}X_{\sigma_4}, X_{\sigma_4}; \lambda) \\ F_{\sigma_1}(X_{\sigma_4}, X_{\sigma_4}, -\frac{b}{b+c}X_{\sigma_2} - \frac{b^2}{c(b+c)}X_{\sigma_4}, X_{\sigma_4}; \lambda) \end{pmatrix},$$

with F_{σ_1} satisfying the above restriction. Assuming from this point on that $\lambda \in \Lambda := \mathbb{R}$, one can quite easily translate the restriction on F_{σ_1} into a set of equations for its Taylor series coefficients (up to any desired order), that we do not present here. The analysis of the equation $G(X_{\sigma_2}, X_{\sigma_4}; \lambda) = 0$ now proceeds as in the previous examples.

For instance, it is clear that $G_1 = G_2$ when we put $X_{\sigma_2} = X_{\sigma_4}$ (corresponding to partial synchrony). Setting $X_{\sigma_2} = X_{\sigma_4}$ in the equation $G_1 = 0$ then gives that $X_{\sigma_2} = X_{\sigma_4} = 0$ or $X_{\sigma_2} = X_{\sigma_4} \sim \lambda$, under generic conditions on the Taylor coefficients of F_{σ_1} .

To find non-synchronous solutions, one may observe that the equation $\frac{G_1 - G_2}{X_{\sigma_2} - X_{\sigma_4}} = 0$ generically leads to a relation of the form $X_{\sigma_2} = X_{\sigma_2}(X_{\sigma_4}, \lambda)$. Substituting this relation in the equation $G_2 = 0$ then yields a solution

branch in which $X_{\sigma_2} \sim \lambda$ and $X_{\sigma_4} \sim \lambda$. Furthermore, doing the calculation explicitly one finds that $X_{\sigma_2} - X_{\sigma_4} \sim \lambda^2$ generically. In particular, this branch is not partially synchronous. Summarizing, we find the following local branches of steady state solutions:

$$\begin{aligned} X_{\sigma_1} &= X_{\sigma_2} = X_{\sigma_3} = X_{\sigma_4} = X_{\sigma_5} = 0, \\ X_{\sigma_2} &= X_{\sigma_4} = X_{\sigma_5} \sim \lambda, X_{\sigma_1} = X_{\sigma_3} = -\frac{\textcolor{blue}{b}}{\textcolor{red}{c}} X_{\sigma_2} \sim \lambda \text{ and} \\ X_{\sigma_2} &= X_{\sigma_5} \sim \lambda, X_{\sigma_4} \sim \lambda, X_{\sigma_1} = X_{\sigma_3} = \frac{-b}{b+\textcolor{red}{c}} X_{\sigma_2} - \frac{b^2}{\textcolor{red}{c}(b+\textcolor{red}{c})} X_{\sigma_4} \sim \lambda, \end{aligned}$$

where $X_{\sigma_2} - X_{\sigma_4} \sim \lambda^2$ for the last branch. The identification $X_{\sigma_1} = X_{\sigma_3} = x_{v_3}$, $X_{\sigma_2} = X_{\sigma_5} = x_{v_1}$ and $X_{\sigma_4} = x_{v_2}$ then yields the results on network **C** reported in Table 2.1.

Under generic conditions on the response function $f = f(X_{\sigma_1}, X_{\sigma_2}, X_{\sigma_3}; \lambda)$ of networks **A**, **B** and **C**, Lyapunov-Schmidt reduction at a synchrony breaking bifurcation leads to a reduced bifurcation equation $F(X; \lambda) = 0$ that satisfies all the nondegeneracy conditions required of a generic equivariant bifurcation. This fact can be checked by performing the Lyapunov-Schmidt reduction explicitly, and such an analysis proves that the asymptotics displayed in Table 2.1 is correct. Not surprisingly, the explicit Lyapunov-Schmidt reduction requires a long analysis as well. For now, it is enough to remark that “generic hidden symmetry considerations” correctly predict the content of Table 2.1.

Information on the stability of the bifurcating branches can not be obtained from Lyapunov-Schmidt reduction, but can be revealed with techniques like centre manifold reduction. We are currently developing this technique for dynamical systems with semigroup symmetry, so we shall not prove any of the statements on stability that were made in Section 2.3.

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